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Dr. LEE, President, in the Chair.

J. Norman Lockyer, Esq., Wimbledon,
Rev. Edward Crofton, Heene, Sussex; and
J. J. Cole, Esq., 24 Essex Street, Strand,
were balloted for and duly elected Fellows of the Society.

M. Charles Delaunay, Paris, was balloted for and duly elected an Associate of the Society.

Collected Works of Gauss.

The Academy of Sciences of Göttingen announce the publication of the Collected Works of Gauss, including the manuscripts left at his decease. The Works will appear in seven volumes, quarto, under the titles,—

I. Disquisitiones Arithmeticæ.
II. Hohëre Arithmetik.
III. Analysis.
IV. Geometrie und Methode der kleinsten Quadrate.
V. Mathematische Physik.
VI. Astronomie.
VII. Theoria Motus Corporum Cœlestium;

the *Theoria Motus* to appear when the copyright interest therein has expired. The contents of the several volumes are given in No. 1348 of the *Astronomische Nachrichten.*

The Works are to be published by subscription, which may be either for the whole or for the separate volumes; the subscription price to be hereafter fixed, but to be at about the rate of four thalers for a volume of from fifty to sixty sheets. Subscriptions may be addressed, post-free, "An das Secretariat der Königlichen Gesellschaft der Wissenschaften zu Göttingen." On completion, the Works will be sold in the ordinary course, at the rate of six or seven thalers per volume. The impression has begun, and will be carried on so that the first six volumes may at latest appear within five years.

On the Perturbations of Uranus and the Mass of Neptune. By Truman Henry Safford, Assistant at the Observatory of Harvard College.

In the determination of the mean distance of the satellite of *Neptune*, and the consequent evaluation of that planet's mass, the faintness of the satellite has proved a great difficulty; so that different observers have differed systematically *inter se*, although each one is quite consistent with himself.

For instance, the mass of *Neptune*, from O. Struve's observations of the satellite, comes out $M = \frac{1}{14491} \pm 0{\cdot}02431\ M$, while Prof. G. P. Bond has deduced $M = \frac{1}{19400}$ from his father's and his own observations.

The perturbations of *Uranus*, by means of which *Neptune* was originally discovered, are, of course, adequate to determine its mass; and it would naturally be expected that the probable error of such a result would be much less than the difference between the numbers cited above.

Prof. Peirce, employing, I believe, the same observations as Leverrier did in his "Recherches sur les Mouvements de la Planète *Herschel* (dite *Uranus*)," has deduced (*Ast. Nach.*, No. 637) the result that, in general, a mass of *Neptune* nearly $\frac{1}{20000}$ will represent the motions of *Uranus* considerably nearer than a value more closely approaching to Struve's.

I was led to take up this question partly because it had been, some years ago, made the subject of a prize question by the Haarlem Society of Sciences (which, so far as I know, was not answered within the specified time), and partly because Struve's mass of *Neptune* seems to have been adopted by some high authorities; by Leverrier as, at least, a provisional value.

The perturbations of *Uranus* produced by *Neptune*, assuming $\frac{1}{19400}$ as the disturbing mass, were computed by the method of mechanical quadratures. These were then added to the perturbations of heliocentric longitude derived from Leverrier's theory, and the theory thus computed (so far as heliocentric longitude is concerned) was compared, for one date in each year, with the results from Bouvard's Tables contained in the *Greenwich Planetary Reductions*, and with the *Nautical Almanac* since 1836 inclusive, to 1858.

Errors of these latter theories in heliocentric longitude were obtained from the *Greenwich Planetary Reductions*, and the *Results of the Greenwich Observations*, 1836–1858; and thus the errors in heliocentric longitude of the present provisional theory were obtained.

It is proper to mention here, that Bradley's observation of 1753 was omitted here, as it seemed desirable to found the result on modern observations, using the ancient ones as a confirmation; also that where the error of radius vector in Bouvard's Tables was found sensible, observations in the neighbourhood of opposition were used.

The Paris observations (1801 to 1819), the Greenwich (1831 to 1835), and the Konigsberg observations (1814 to 1846), were also used; the first series, as reduced by Leverrier, in the *Annales de l'Observatoire Impériale*, and the remainder after Flemming's (*Ast. Nach.* No. 755 *et seq.*), and Applebaum's (*Königsberg Observations*, vol. xxxii. p. 210 *et seq.*) reductions. It is known that the observations at these Observatories were made chiefly near opposition; and, indeed, Flemming has computed (with a few mistakes) the opposition as it is derived from observations made near it for a good many years.

Of course, every effort has been made to put all these reductions in the state in which they would have been if uniform star-places and other elements had been used in making them.

The results were found to indicate with some degree of probability, by a mere inspection, that the disturbing mass would need but little change. In fact, a preliminary solution of a set of twenty-six equations obtained by equating the residual errors of this first theory (taking the means of three years together) to functions of unknown variations of the elements of *Uranus* alone, represented these equating with final remainders of less than $4''$ in every case.

It seemed expedient, however, to introduce a variation of *Neptune's* mass as an unknown quantity; and after another preliminary solution, whose object was to obtain means for combining (as before) the observations of three years together, the weights were assigned to different series of observation, according to a somewhat arbitrary distribution. A final solu-

tion then gave the mass of *Neptune* $M = \frac{1}{20039}$ with a probable error in the denominator $\pm$ 295.

The increase of *Neptune's* mass to $\frac{1}{14491}$ would have the effect of nearly quadrupling the mean error of observation; and would, in fact, leave intolerable discrepancies in several places.

An investigation now made into the geocentric longitudes derived from observations made before *Uranus* was actually recognised as a planet, gave eminently satisfactory results. The following is the comparison, with the observations used by Leverrier, p. 138 of his *Recherches*, "to form the first twelve equations, which are all ancient:"—

Year.	No. Obs.	Observer.	C — O.
			"
1690	1	Flamsteed	— 5·0
1712	1	,,	+ 18·1
1715	2	,,	+ 5·9
,,	1	,,	+ 0·8
1750	1	Lemonnier	+ 4·0
,,	1	,,	— 1·6
1753	1	Bradley	+ 0·2
1756	1	Mayer	+ 0·9
1764	1	Lemonnier	— 0·4
1768	2	,,	— 2·3
1769	6	,,	— 5·3
1771	1	,,	+ 5·0

Flamsteed's observation made in 1712 is uncertain, on account of the rate of the clock. It is isolated about $2\frac{1}{2}$ hours from the comparison stars; and, during an interval of twenty or more days, for which there are only one night's observations, the clock was often very irregular.

Grouping the observations, as Leverrier has done (p. 239 of the Memoir above cited), we shall have as residual errors,—

	Error.	Leverrier's Limit.
	"	"
1690	— 5	25
1712–15	+ 8	15
1750	+ 1·2	15
1753–56	+ 0·55	10
1768–69	— 4·55	10

All these observations are, therefore, represented much within the limits given by Leverrier, p. 240 of his *Recherches*.

On the Proper Motion of Sirius in Declination. By Truman Henry Safford, Assistant at the Observatory of Harvard College.

It is well known that *Sirius* exhibits irregularities of proper motion, both in Right Ascension and Declination. Bessel (*Ast. Nach.* Nos. 514–16), and Dr. C. A. F. Peters, in a Memoir, "Ueber die eigene Bewegung der *Sirius* (Habilitationsschrift), Königsberg, 1851," also republished in vol. xxxii. of the *Astronomische Nachrichten*, have investigated the variations of proper motion in Right Ascension. The hypothesis which these distinguished astronomers have adopted is, that the motions are performed around some centre of gravity, which is not at *Sirius* itself. This involves the idea of a dark body of considerable mass, whose attraction is exerted to draw *Sirius* from its hypothetical motion in a straight line. Dr. Peters does not, as I understand, insist that the dark body is of greater mass than *Sirius* itself; so that it may (possibly) be of planetary character.

The object of the present discussion is merely to show the character of the irregularities in Declination, and to examine whether they are reconcilable with the results derived from the Right Ascensions. The quantity which I have used in this investigation is the deviation of *Sirius* in Declination from the *Tabulæ Regiomontanæ*, less mean of the corresponding deviation for the four stars *Spica*, α^2 *Libræ*, *Antares*, and α^2 *Capricorni*. These have the advantage over absolute declinations, that there is thus a possibility of avoiding a portion of the systematic errors to which most astronomical determinations are exposed. The mean of the Declinations of the four comparison stars is very nearly that of *Sirius*.

Bessel's Declinations are (it is well known) more than a second further south than other determinations, at a South Declination equal to 16°; and in a matter of such delicacy it becomes necessary to scrutinise the observations quite closely.

Without a new reduction of Piazzi's and Pond's observations, it would be difficult to use them as fully authentic. The chief cause of disturbance seems to be the fact that Piazzi and Pond both used Bradley's factor for the thermometric variation of the refraction; and this value is largely erroneous. If *Sirius* is observed at a mean temperature different from that at which the four stars above mentioned are observed, an error is thus introduced. The refraction being large, it becomes of too great influence.

The effect (probably) of this error in the factor of the thermometric correction may be seen on comparing Olufsen's reduction of Pond's observations of 1822 with Pond's own. As α^2 *Capricorni* is not contained in Pond's reduction, we will (at

this period) omit it in considering Olufsen's. We find that by Pond's own reduction of his observations of 1822, *Sirius* is 1″·68 further north than the *Tabulæ Regiomontanæ* make it; that *Spica*, α^2 *Libræ*, and *Antares*, are (in the mean) 4″·55, and, therefore, that *Sirius* is 2″·87 too far south, as compared with these three stars. But, by employing Olufsen's reduction, *Sirius* is shown to be 0″·33 too far north, and the three other stars in the mean 0″·95; so that *Sirius*, as compared with these three, is 0″·62 too far south.

The comparisons here given have been taken from Bessel's table (*Ast. Nach.* No. 73, p. 7).

The numbers which have been used are as follows, nearly:

			q'.
			″
Bradley	1755		−0·7
Piazzi	1805		(−0·5)
Pond	1813		(+0·6)
Bessel	1820		−0·4
Pond	1822	(Olufsen)	−1·5
Struve	1824		−1·6
Pond	1826		(−2·1)
Argelander	1830		−1·0
Taylor*	1832		−1·7
Pond	1833		(−2·6)
Henderson*	1833		−1·5
Airy	1834		−2·4
Maclear*	,,		−1·7
Henderson	1837		−0·8
Airy	1838		−0·4
Busch	,,		−0·4
Bessel	1843		+1·1
Airy	1844		+1·9
,,	1850		+2·1
Moesta*	1855		+0·8
Airy	1856		+1·5

* From observations near and south of the equator.

It may seem singular that the position of the *Fundamenta* for 1755, and Bessel's for 1820, should seem to differ at all from the *Tabulæ Regiomontanæ*, when the latter were calculated from the former. The numbers given above, for those Catalogues, are merely the corrections necessary, on account of the alteration in mutation, rendered necessary by more modern observations. Corrections have been applied also to Olufsen's reduction of Pond (−0″·26), to Argelander (+0″·40), and to Busch (−0″·39); these being the excess of the correction

taken from Table IV. on p. xxvi. of Struve's *Positiones Mediæ* for *Sirius* over the similar correction for the four stars compared.

Our table includes, then, the deviation from the *Tabulæ Regiomontanæ* of the Declination of *Sirius*, as observed by different observers, with (in general) different instruments, less the mean of the similar corrections for *Spica*, a^2 *Libræ*, *Antares*, and a^2 *Capricorni*; corrected in the cases above specified for error of mutation.

On referring to Dr. Peters' memoir, we shall at once find that, if his hypothesis be adopted, the numbers given above will be represented by the formula

$$q' = b' + f' + c'(t - 1800) + g' \cos u + h' \sin u,$$

where b' denotes the correction of the Declination for 1800 of the centre of gravity of the system, c' the correction of Annual Proper Motion; each of these quantities to be applied to the *Tabulæ Regiomontanæ* values for *Sirius* itself, to get declinations in general accordance with Bessel's other declinations; g', h', being constants depending on the elements of the orbit of *Sirius* around the centre of gravity, and $f' = -g'e$, where e is the excentricity; u being also the excentric anomaly of *Sirius* in its orbit.

If, then, we adopt Peters' Elements V.,

Passage of the Lower Apside	1791·431
Mean Yearly Motion	7°·1865
Excentricity (e)	0·7994

we shall have

$$u - e \sin u = 7^{\circ}{\cdot}1865\,(t - 1791{\cdot}431),$$

t being the date in years, and the predicted values of $\sin u$ $\cos u$ can be easily computed. But, in the first place, it must be seen whether any hypothesis of uniform motion will represent the observations, excluding Bradley's of about 1755. It is found that a correction $-4''{\cdot}18 + 10''{\cdot}30\left(\frac{t-1800}{100}\right)$ will materially lessen the errors. This correction, however, requires that Bessel's Bradley's place for 1755 be erroneous by $8''{\cdot}7$; and that this is impossible is shown by the near agreement of Lacaille's and Mayer's places for 1750 and 1756 with the *Tabulæ Regiomontanæ*.

But the actual formula deduced from the observations by the substitution of Peters' hypothesis is

$$-0''{\cdot}56 + 0''{\cdot}0202\,(t - 1800) + 1''{\cdot}47 \sin u + 0''{\cdot}51 \cos u,$$

and this leaves, as residual errors,

			C − O.	
			″	
Bradley			0·0	
Piazzi	1805		(+0·8)	
Pond	1813		(−0·6)	
Bessel	1820		−0·6	
Pond	1822	(Olufsen)	+0·3	
Struve	1824		+0·4	
Pond	1826		(+0·8)	
Argelander	1830		−0·5	
Taylor	1832		+0·2	
Henderson	1833		0·0	
Pond	1833		+1·1	(14 observations.)
Airy	1834		+1·0	
Maclear	1834		+0·3	
Henderson	1837		−0·4	
Airy	1838		−0·5	
Busch	1838		−0·4	
Bessel	1843		+0·5	
Airy	1844		0·0	
,,	1850		−0·4	
Moesta	1855		+0·5	
Airy	1856		−0·4	

There can be hardly any doubt, then, that Bessel's and Peters' hypothesis does actually represent the observed Declinations of *Sirius*, although the evidence on which it has been previously urged depends entirely on the Right Ascensions.

It may be noticed that Calandrelli's statement, that the *Greenwich Twelve-Year Catalogue* for 1845 (1844) is materially in error, is at once refuted by the almost perfect agreement given above. The phenomena which *Sirius* presented about 1841, that being according to Peters (confirmed by the results above) the time of the passage of the lower apside, are sufficiently remarkable; rapid variations both in Right Ascension and Declination were then taking place.

Extract of a Letter from Mr. A. Auwers to the Rev. R. Main, dated Königsberg, 1862, February 21, on the Irregularity of the Proper Motion of Sirius, and on a Missing Nebula.

(*Communicated by the Rev. R. Main.*)

For some months I have been attempting to represent the observed Declinations of *Sirius* from 1755 to 1859, in which a

periodical change of proper motion is very conspicuous, by a formula grounded on the elements of Peters; and I found the correction q' of the declinations of the *Tabulæ Regiomontanæ*, relatively to β *Orionis* and α *Hydræ*, from thirty-one equations,

$$q' = + 0''{\cdot}10 + 0''{\cdot}0158\,(t - 1800) + 1''{\cdot}536 \sin u - 0''{\cdot}025 \cos u,$$

where

$$u - 0''{\cdot}7994 \sin u = 7^{\circ}{\cdot}1865\,(t - 1791{\cdot}431),$$

The values of q' from the *Greenwich Six-Year Catalogue*, and from the observations from 1854 to 1859, are

$$- 0''{\cdot}14,\ + 2''{\cdot}36,\ + 2''{\cdot}51,\ + 2''{\cdot}65;$$

the elements give for these epochs

$$- 0''{\cdot}66,\ + 2''{\cdot}14,\ + 2''{\cdot}36,\ + 2''{\cdot}03,$$

and therefore completely justify your defence of the Greenwich declinations against the objections of Calandrelli.

It has only very recently been known to me that Mr. Safford has made a similar investigation, using, however, different stars of comparison. If the corrections q' of the declinations of *Sirius* be deduced from the formula given above, and from that communicated in No. 28 of the *Astronomical Notices*, we obtain: —

According to Safford,

$$q_{,}' = - 1''{\cdot}00 + 0''{\cdot}0104\,(t - 1800) + 1''{\cdot}47 \sin u + 0''{\cdot}51 \cos u$$

According to my calculation,

$$q_{,}' = - 0''{\cdot}60 + 0''{\cdot}0003\,(t - 1800) + 1''{\cdot}536 \sin u - 0''{\cdot}025 \cos u$$

a satisfactory agreement with reference to the materials employed. This in fact is not unexceptionable, the probable error of a determination of declination being, according to Safford, $\pm\ 0''{\cdot}30$, and according to me, $\pm\ 0''{\cdot}32$, while at the same time I have found for *Procyon* an orbit, which represents the observed places with the probable errors $\pm\ 0''{\cdot}17$. The reason of the lesser agreement of the Declination of *Sirius* arises plainly from the circumstance that this star for European Observatories culminates at too low an altitude. For a new determination of the elements of its orbit from all the observed Right Ascensions and Declinations, which I contemplate undertaking, determinations of southern Observatories would be very desirable. Is it known to you whether the observations of the Cape of Good Hope since 1834 are reduced?

I observed several times in February and March, 1858, the interesting Nebula found by Hind in 1852, and observed by D'Arrest and Breen in 1855 and 1856, and finally missed by D'Arrest in October 1861. In 1858 I found it to be fainter than it ought to have been, according to the description of D'Arrest in 1855. At that time I saw it with a telescope of $4\frac{1}{3}$ inches aperture, but, in January and September 1861, I could not find it with the 6-inch heliometer, nor have I been able to find it after repeated searches during the last five months. The variability of the accompanying star I remarked in November 1861, as I found it to be of the 11–12 magnitude, while in 1858 it was of the 10th magnitude. At present it appears to be getting somewhat brighter, and is of the 11th magnitude.

Extract of a Letter from Mr. A. Auwers to the Rev. R. Main, dated Königsberg, 1862, March 21, on the Orbit of Procyon, and on the Positions of the Radcliffe Catalogue.

(*Communicated by the Rev. R. Main.*)

I am much obliged for your answer to my letter of the 21st of February. Learning from your letter that the results of my researches on variable Proper Motions may perhaps be interesting to you, I add to the formula for correcting the tabular Declinations of *Sirius*, given in my preceding letter, also the elements of the orbit of *Procyon*, derived from all available observations of Right Ascension and Declination, made between 1750 and 1860:—

Epoch of minimum in R.A.	$=T$	$1795{\cdot}5676$	Probable error	$\pm 0^{y}{\cdot}4457$
Time of revolution	$=U$	$39^{y}{\cdot}972$	,, ,,	$\pm 0^{y}{\cdot}4043$
Mean yearly motion	$=n$	$9^{\circ}{\cdot}00634$	,, ,,	$\pm 0^{\circ}{\cdot}09110$
Radius of circle	$=r$	$1''{\cdot}0525$	,, ,,	$\pm 0''{\cdot}0275$

There is not the least trace of ellipticity indicated by the observations. The corrections to be applied to the Right Ascension of Wolfers' *Tabulæ Reductionum*, relative to the mean of the stars *Aldebaran*, *Rigel*, *α Orionis*, *Pollux*, *α Hydræ*, and *Regulus*, and to the Declination of the same tables, relative to *α Ceti*, *α Orionis*, *α Serpentis*, *γ*, *α*, *β Aquilæ*, and *α Aquarii*, are the following:—

$$d\alpha = +0^{s}{\cdot}0033 - 0^{s}{\cdot}001192\,(t-1830) - 0^{s}{\cdot}0705\cos n\,(t-T),$$

$$d\delta = +0''{\cdot}201 + 0''{\cdot}00931\,(t-1830) + 1''{\cdot}853\,\sin n\,(t-T).$$

The mass of the dark companion of *Procyon* is probably not less than three-quarters of the mass of the Sun.

Variations have also been supposed to take place in the proper motions of *α Virginis*, *α Hydræ*, and *β Orionis;* but in all these cases the results of my researches are entirely negative, showing with evidence that there exist no variations in the motions of these stars.

The reason of my again taking the liberty to write to you is a paper "On the Positions of the Radcliffe Catalogue," in No. 9 of vol. xxi. of the *Monthly Notices*. It has not been before now that I have learnt that Mr. Safford calls attention by this paper to a rather large discrepancy between a formula, given by me in No. 1300 of the *Astronomische Nachrichten*, for correcting the Declinations contained in the vol. i.–xiv. of the *Radcliffe Observations*, so as to make them agree with Wolfers' *Tabulæ Reductionum*, and the corrections to be applied to these Declinations according to the late Mr. Johnson's investigation in vol. xv.

The formula has been derived, as it is stated in the *Astronomische Nachrichten*, from Mädler's comparisons (*Dorpat Observations*, xiv.), and represents them very accurately between the Declinations 40° and 80°. But now the definitive Declinations of the Radcliffe Catalogue having been substituted to the provisionally adopted ones of the *Observations*, I have thought it advisable to undertake a direct comparison of the first with some other Catalogues, the relations of which to the *Tabulæ Reductionum* are accurately known.

There occur 231 stars of the Radcliffe Catalogue in Argelander's Abo Catalogue reduced to 1845, and corrected for the difference between the constants of Nutation adopted in the construction of the Catalogue and the computation of the Proper Motions and the true value $=9''\cdot223$; the correction to be applied to the Declinations being $+0''\cdot244 \sin(\text{R.A.}-0^h36^m)$, and to the Proper Motions $0''\cdot0081 \sin(\text{R.A.}+0^h55^m)$. Argelander's Declinations differ from Johnson's (corrected, where necessary, according to the introduction of the Catalogue) by the following quantities, A—R, to which I have also added the differences W—A between the *Tabulæ Reductionum* and the Abo Catalogue, reduced to the year 1845, and the differences W—R, resulting from the addition of the numbers A—R and W—A.*

Decl.	A—R.	Stars Comp.	W—A.	W—R.
°	″ ″		″	″
−16·6	+1·45 ±0·15	17	−0·72	+0·73
− 4·9	+1·10 ±0·19	11	−0·63	+0·47
+ 4·9	+0·35 ±0·13	24	−0·53	−0·18
+14·9	+0·73 ±0·18	12	−0·41	+0·32

* Four stars presenting large differences have been excluded in taking the means.

Decl.	A—R.		Stars	Comp.	W—A.	W—R.
°	″	″			″	″
+25·9	−0·41	±0·21	9		−0·25	−0·66
+35·9	−0·50	±0·24	7		−0·09	−0·59
+41·7	−0·12	±0·10	38		−0·01	−0·13
+48·8	+0·12	±0·11	31		+0·10	+0·22
+53·3	+0·53	±0·12	27		+0·18	+0·71
+62·5	+0·45	±0·14	19		+0·26	+0·71
+69·8	+0·67	±0·15	17		+0·27	+0·94
+76·7	+0·92	±0·22	8		+0·24	+1·16
+84·1	+0·99	±0·22	7		+0·15	+1·14

In a similar way I have found by 408 stars common to the Radcliffe Catalogue and Struve's *Positiones Mediæ* (including 93 already made use of in the preceding investigation),

Decl.	S—R.		Stars	Comp.	W—S.	W—R.
°	″	″			″	″
−15·4	+1·81	±0·24	10		−0·82	+0·99
+ 2·0	+0·79	±0·16	19		−0·83	−0·04
+16·2	+0·52	±0·17	20		−0·39	+0·13
+30·6	−0·49	±0·18	18		−0·23	−0·72
+41·1	+0·08	±0·09	70		−0·25	−0·17
+48·6	+0·29	±0·10	53		−0·21	+0·08
+55·1	+0·51	±0·08	80		−0·09	+0·42
+62·4	+0·48	±0·10	56		+0·11	+0·59
+69·3	+0·77	±0·12	37		+0·18	+0·95
+76·6	+0·53	±0·15	25		+0·14	+0·67
+83·7	+0·89	±0·17	20		+0·02	+0·91

The mean of the two nearly agreeing systems of values W—R becomes, for the circumpolar stars,

Decl.	W—R.	Stars.	Formula.	C—O.
°	″		″	″
41·4	−0·15	108	−0·16	−0·01
48·7	+0·13	84	+0·16	+0·03
55·2	+0·49	107	+0·41	−0·08
62·4	+0·62	75	+0·16	+0·04
69·5	+0·95	54	+0·84	−0·11
76·6	+0·79	33	+0·95	+0·16
83·8	+0·97	27	+1·02	+0·05

Assuming the Weights of these numbers nearly proportional to the numbers of the stars compared, I have derived from them the following formula:—

$$W-R = -2''\cdot53 + 3''\cdot59 \sin \delta.$$

The column headed C—O contains the small remaining differences between this formula and the observed values. Tabulating the values of the formula with changed sign, we receive the following

Corrections of the N.P.D. of the Radcliffe Catalogue.

N.P.D.		Observ. vol. i.–xiv. Johnson.	Formula.
0°	−1″·06	+0″·92	−0″·14
5	−1·05	+0·97	−0·08
10	−1·01	+1·03	+0·02
15	−0·94	+1·10	+0·16
20	−0·84	+1·15	+0·33
25	−0·72	+1·23	+0·51
30	−0·58	+1·28	+0·70
35	−0·41	+1·34	+0·93
40	−0·22	+1·38	+1·16
45	−0·01	+1·43	+1·42
50	+0·22	+1·46	+1·68

In the vicinity of the pole, therefore, the provisionally adopted Declinations of the *Observations* agree far better with the *Tabulæ Reductionum* than the corrected Declinations of the Catalogue; but in Polar Distances greater than 25°, the opposite takes place.

The differences given by the Southern stars are somewhat irregular and less sure; the number of these occurring in the Catalogue being not at all great, I have thought it sufficient to join the values found between 54° and 106° N.P.D. to the circumpolar formula by drawing a simple curve, from which I have taken the following corrections of the N.P.D. of the Catalogue:—

N.P.D.		N.P.D.	
55°	+0″·5	85°	−0″·1
60	+0·6	90	−0·3
65	+0·6	95	−0·5
70	+0·5	100	−0·7
75	+0·3	105	−0·8
80	+0·1	110	−1·0

I may also mention that the differences between the Radcliffe Declinations and Argelander's in the hours 3—11 of R.A. are on the whole smaller (about 0″·5) than in the other hours; but in the comparisons with Struve, no such variation being shown, I have not had regard to it. The probable error of a difference A—R, by one star, is found from the circumpolar

stars to be $\pm 0''\cdot622$, and that of a difference S—R is found to be $\pm 0''\cdot743$. The probable error of a Declination of the Abo Catalogue being $\pm 0''\cdot27$ and that of the reduction to 1845 being $\pm 0''\cdot16$, the probable error of a Declination of the Radcliffe Catalogue becomes, from the comparisons with Argelander, $\pm 0''\cdot54$.

Observatory, Königsberg, March 21, 1862.

Observations of Transit of Mercury, Solar Eclipse, and Occultation of Venus. By M. C. Bulard, Director of the Observatory of Algiers.

(*Communicated by the Astronomer Royal.*)

Transit of Mercury, 1861, *November* 11, *observed at the Observatory of Algiers.*

Egress, exterior* contact.

	h	m	s	
	21	31	50·673	Algiers Mean Time,
or	21	28	28·073	Paris Mean Time.

Solar Eclipse of 1861, *December* 31.

The disturbed state of the country prevented M. Bulard from reaching the line of totality. The following observations were made at Ouargla:—

Latitude 31° 57′ 9″·0 N.; Longitude (approximate) $2^m\ 5^s\cdot33$ E. of Greenwich.

	h	m	s	
First Contact at	2	18	3·8	Ouargla Mean Time.
Last Contact at	4	41	16·1	,, ,,

Occultation of Venus by the Moon, 1862, *February* 1, *observed at Laghouat.*

Latitude 33° 48′ 10″ N.; Longitude (approximate) $11^m\ 33^s$ E. of Greenwich.

	h	m	s	
First Contact at	6	55	10·558	(Apply. Laghouat M. T.).
Disappearance of Left Horn of Venus	6	55	58·515	,,
Disappearance of Right Horn ...	6	56	0·011	,,

(as seen in an inverting telescope).

* There are the words of M. Bulard; but his diagram shows that it was internal contact. G. B. A.

Ephemeris of the Long Period Variable Stars for 1862. By N. R Pogson, Director of the Madras Observatory.

The initials in the last column indicate respectively, Messrs. Argelander, Baxendell, Krüger, Pogson, Schönfeld, and Winnecke.

Star.	Probable Mag.	Mean Place, 1861. R.A. h m s	N.P.D. ° ′	Times of Maxima.	Authority.
Piscium	9·5	0 24 46	76 14	April 24, Sept. 14	S.
Cassiopeiæ	2·0	0 32 35	34 14	March 20, June 7, Aug. 25, Nov. 12	A.
Piscium	9·0	1 10 15	81 49	Possibly in August. Very uncertain	—
Piscium	7·5	1 23 25	87 50	Possibly in May. Uncertain ...	—
Arietis	8·0	2 8 10	65 36	Jan. 15, July 17	B.
Ceti	2·0	2 12 17	93 37	July 3	A.
Tauri	8·0	4 20 38	80 9	May 7	W.
Tauri	10·0	4 21 32	80 22	March 11	W.
Orionis	9·0	4 51 22	82 5	June 19	W.
Orionis	1·0	5 47 36	82 37	January and July. Uncertain ...	—
Geminorum	7·3	6 58 56	67 5	Jan. 25	P.
Can. Min.	8·0	7 1 0	79 46	April. Uncertain	—
Can. Min.	7·5	7 25 7	81 23	July 23	W.
Geminorum	9·2	7 34 38	66 13	August 14	P.
Geminorum	8·9	7 40 54	65 55	May 13	P.
Geminorum	9·1	7 46 48	67 38	Jan. 21, April 28, Aug. 2, Nov. 7	P.
Cancri	6·0	8 8 51	77 51	January 17	B.
Cancri	9·0	8 27 45	70 37	July 3	W.
Hydræ	8·5	8 46 16	86 24	January 22, October 5 ...	W.
Cancri	12·0	8 48 40	69 37	At minimum, December 26 ...	W.
Hydræ	6·5	8 48 51	98 37	February 22, December 11 ...	W.
Leonis	5·0	9 40 2	77 55	Oct. 13. Minimum (10·0) May 29	P.
Urs. Maj.	7·0	10 34 41	20 29	April 29. Minima (13·0) Jan. 6, Nov. 4	P.
Leonis	9·0	11 3 36	83 47	February 9, August 28 ...	W.
Comæ	8·0	11 57 4	70 26	October 1	W.
Virginis	6·5	12 31 24	82 14	Jan. 1, May 27, Oct. 19 ...	A.
Urs. Maj.	7·5	12 37 48	28 8	April 10, Nov. 22. Minima (12) Jan. 2, Aug. 15	P.
Virginis	7·5	12 44 0	83 41	July 28	W.
Virginis	7·0	13 20 36	92 28	March 16, November 23 ...	P.
Hydræ	4·0	13 22 4	112 33	Not until February 1863 ...	B.

Star.	Probable Mag.	Mean Place, 1861. R.A. h m s	N.P.D. ° ′	Times of Maxima.	Authority
S Virginis	6·0	13 25 42	96 28	May 21. Minimum (11·3) Nov. 17	P.
R Boötis	8·0	14 31 1	62 39	February 16, September 29 ...	W.
S Serpentis	8·0	15 15 7	75 11	March 5	A.
R Coronæ	6·2	15 42 49	61 25	About November. Very irregular	B.
R Serpentis	6·5	15 44 15	74 26	August 29	A.
R Libræ	9·0	15 45 40	105 49	April. Uncertain	P.
R Herculis	8·5	15 59 56	71 15	Oct. 24	B. & W.
R Scorpii	9·0	16 9 19	112 35	April 22. Uncertain ...	P.
S Scorpii	9·5	16 9 20	112 33	April 19. Very irregular ...	—
S Ophiuchi	9·3	16 26 12	106 52	August 5	—
S Herculis	7·5	16 45 32	74 49	September 15	B.
R Ophiuchi	8·0	16 59 44	105 54	June 25	P.
α Herculis	3·0	17 8 16	75 27	{Mar. 11, May 16, July 21, Sept. 26, Dec. 1 ... ...}	A.
T Herculis	7·9	18 3 48	59 0	March 16, August 23 ...	K.
R Scuti	5·0	18 40 1	95 50	March 21, May 31, Aug. 11, Oct. 26	A.
13 Lyræ	4·3	18 51 4	46 14	{March 18, May 3, June 18, Aug. 3, Sept. 18, Nov. 3 ... ...}	B.
R Aquilæ	6·5	18 59 38	81 59	May 13	A.
R Sagittarii	8·2	19 8 28	109 33	May 19	P.
S Sagittarii	10·5	19 11 14	109 16	December. Very uncertain ...	P.
R Cygni	8·0	19 33 4	40 7	Nov. 14. Minimum (14·5) June 12	P.
χ Cygni	5·0	19 45 11	57 27	June 10	A.
R Capricorni	9·5	20 3 28	104 41	September 25	W.
R Sagittæ	8·4	20 7 40	73 42	{Minima (10·5) May 6, July 16, Sept. 25, Dec. 5 ... ...}	B.
S Delphini	8·2	20 36 38	73 25	May 4	B.
U Capricorni	10·5	20 40 22	105 18	April. Uncertain	P.
R Vulpeculæ	8·0	20 58 10	66 44	March 8, July 16, Nov. 23 ...	W.
T Capricorni	9·0	21 14 13	105 45	July 26	S.
R Pegasi	8·5	22 59 37	80 14	April 28	A.
R Aquarii	7·0	23 37 15	106 3	June 28	—
R Cassiopeiæ	6·0	23 51 18	39 23	Nov. 15. Minimum (13) May 30	P.

The preceding Ephemeris has been hastily prepared, chiefly from the elements of variation employed for the last two years.

Mr. Baxendell has as usual favoured me with the maxima of the stars bearing his initial. The minima of the Short Period Variable Stars shall follow by the next mail.

One star, S *Sagittarii*, was discovered at Dr. Lee's Observatory shortly before my departure from England, and has been re-observed here this year. Time forbids further details in this letter.

Madras Observatory, 1861, *Dec.* 28.

On the Stars R *Vulpeculæ and* U *Geminorum, and on an Appearance in Venus.* By G. Knott, Esq.

R *Vulpeculæ.* A few months ago I had the honour of submitting to the Astronomical Society a short paper on an unexpected *minimum* of this star; further observation enables me to report a *maximum*, mag. = 8·7, on or about the 31st December last. It will be seen that this result does not accord with the predictions of Mr. Pogson's valuable Ephemeris; by reference, however, to a paper by Dr. Winnecke, in No. 1224 of the *Astronomische Nachrichten*, it appears that the period is subject to considerable irregularity, for while two observed maxima in 1859 gave a period of 129 days, one of 147·35 days was found more satisfactorily to represent the earlier observations, including two by Piazzi in the beginning of the century. It would be premature on my part to say anything decidedly on this point, I will merely add that my own light-curve seems to favour the shorter period.

U *Geminorum.* This remarkable star was due at maximum on Jan. 21; my own observations show that the maximum actually occurred nineteen or twenty days earlier. On turning my Equatoreal to the spot on the evening of Jan. 3d, I found the star of the 9·4 mag., by the next evening it had fallen to 9·6, and on the 11th of the same month it shone as a star of the 11·2 mag. Bad weather and bright moonlight prevented further observation. Unfortunately the maximum was passed before my observations commenced, but we may probably assume the 1st or 2d of January as the date of its occurrence.

Readers of the *Monthly Notices* will remember an interesting paper by Mr. Pogson (*Monthly Notices*, vol. xx., p. 37) containing an account of a series of observations of this star by M. Goldschmidt in November 1859, from which, in combination with one of his own, he deduced a maximum on Nov. 12th, "full 27 days before the time indicated by computation." Starting from this date, and applying Mr. Pogson's period of 96·879 days, I find a maximum due on Dec. 25th, 1861, a result, which, compared with my own observations, seems to

show that, in the interval, the period has not been subject to any startling irregularity.

I take the opportunity of mentioning a recent observation of that curious phenomenon connected with the planet *Venus*, sometimes called the *phosphorescence of the dark side*. On the evening of Jan. 14th, my uncle, Mr. Berry of Liverpool, was examining the planet with a small but very perfect Gregorian reflector of 4 inches aperture, mag. power 160. The wind was high, but the atmosphere very clear, and in repeated intervals of quiet, when the cusps were sharply defined, the unillumined part of the disk shone with a faint light similar in appearance to the *lumière cendrée* in the crescent Moon. In proof of the independence of the observation I may say, that, at the time, it had entirely escaped Mr. Berry's memory that the phenomenon had been remarked by previous observers.

Woodcroft, Cuckfield, March 10th, 1862.

Orbit of ξ Ursæ Majoris, determined from the Observations made between 1819·10 *and* 1860·08. By J. Breen, Esq.

Perihelion	1816·32
Period	63·14 years.
Excentricity	0·3929
Major semiaxis	2″·454
	° ′
Perihelion from node	132 53
Node	97 18
Inclination	52 16

Comparison of Observed and Calculated Places.

Date.	Angle of Position. [O − C]	Distance. [O − C]	Observer.
	° ′	″	
1819·10	+1 4	(+0·991)	Struve
1826·20	+2 28	+0·007	,,
1831·08	−2 7	+0·157	Bessel
1835·41	+3 11	−0·112	Struve
1840·40	+1 36	+0·055	,,
1843·48	−1 5	+0·235	Bessel
1848·32	−0 19	+0·028	J. Breen
1852·20	+1 18	−0·074	Fletcher
1857·40	+1 14	−0·081	,,
1860·08	+0 58	−0·168	Lord Wrottesley.

Compared with the observations of Sir W. Herschel of 1781·97, and the single measurements of 1802·09 and 1804·08, the calculated angles of position are respectively 5° 42′, 5° 17′, and 6° 21′ in defect.

New Determination of the Longitude of the Sydney Observatory, by means of Thirty-three Moon-Culminations at Greenwich and Sydney, and Twenty-three at the Cape of Good Hope and Sydney. By W. R. Scott, Esq., Astronomer for New South Wales.

	h	m	s
Assumed Longitude of Sydney Observatory	10	5	0
Cape of Good Hope ,,	1	13	55

Greenwich Date.	Correction.	Greenwich Date.	Correction.
	s		s
1860, Jan. 4	−18·90	1860, July 3	−10·13
9	16·17	6	2·90
Mar. 1	16·56	30	16·13
Apr. 2	15·79	Oct. 5	15·42
4	12·39	Nov. 1	13·71
6	4·59	2	8·36
7	10·43	22	14·26
9	12·32	Dec. 22	8·87
30	17·65	1861, Jan. 24	14·94
May 1	15·56	28	19·27
2	15·36	Mar. 22	6·56
3	12·98	23	13·02
4	12·96	Apr. 24	15·99
28	17·01	May 17	18·96
June 25	13·34	22	10·78
26	14·33	July 15	−14·48
30	− 7·60		

Cape of Good Hope.

Greenwich Date.	Correction.	Greenwich Date.	Correction.
	s		s
1859, July 9	−22·53	1859, Oct. 5	−17·28
18	9·40	8	33·97
22	8·81	12	12·92
23	9·14	13	15·02
Aug. 11	9·85	14	14·40
12	12·99	17	14·28
13	14·11	Nov. 3	11·89
19	18·15	4	13·91
Sept. 9	20·79	Dec. 5	15·59
13	19·22	8	22·20
17	9·78	14	−14·37
19	−17·53		

	s
Mean of Corrections from Greenwich Observations	−13·26
,, ,, from Cape of Good Hope ,,	−15·75
Mean of all with Equal Weights	−14·21

Resulting Longitude of Sydney Observatory, $10^h\ 4^m\ 45^s{\cdot}79$

On Comet II., 1861, *and on some other Comets.*
By Dr. Mackay.

(*Communicated by Prof. C. P. Smyth.*)

The Rev. Dr. Mackay, Missionary, Chinsurah, Bengal, has communicated through Prof. C. Piazzi Smyth, his (Dr. Mackay's) observations of the Comet II., or the Great Comet, 1861, and his computation of two sets of elements of the orbit, as follows,—

Ascending Node	279° 2′	278° 59′ 30″
Inclination	85° 33′	85° 29′ 32″
Perihelion dist.	·825	·8267
,, long.	249° 38′	249° 51′
,, passage	June 11^{d} 20^{h}	June 11^{d} 23^{h}

Motion direct.

The first was computed from his own observations of 1861, July 5, 10, 15, and was published in India on July 25th; the second one was computed in August, substituting a French observation of June 30th, printed in the *Journal des Debats*, for the first of his own observations.

The observations are remarkable instances of what may be done with "a sextant and an old ship's chronometer." The Doctor writes, "My plan of observing was, to take every available evening a series of the Comet's distances from *Vega* and *Arcturus*, as near as possible to a mean of 9^{h} 5^{m} 33^{s}, Chinsurah Mean Time (3^{h} 12^{m} G.M.T.), and from these to compute the Comet's Right Ascension and Declination. With the found Right Ascension and Declination I computed its distance from a third star, *η Ursæ Majoris*, which I also measured; and it was only when the computed and observed distances agreed, or slightly differed, (say 1′) that I considered the night's work good.

"In October 1858, with the same sextant, but with better eyesight and longer intervals (6th, 18th, and 30th), I found the elements of the Comet of 1858,

	° ′
Perihelion long.	37 4
,, dist.	·5753
,, passage	Sept. 29^{d} 18^{h} 21^{m}
Long. ascending Node	165° 15′
Inclination	63° 39′

Motion retrograde.

"These were published in the *Friend of India*, about a month before receiving the European determination, thus,

Perihelion long.	36° 13′
,, dist.	·579
,, passage	Sept. 29^{d} 23^{h}
Longitude of Node	165° 19′
Inclination	63° 2′

Motion retrograde."

The following is the Doctor's printed local communication, published in the *Friend of India*, July 25, 1861, which, amongst other interesting particulars, contains his identification of the Comet with Comet I., 1781, computed by Mechain.

"With very considerable labour, by three reduced sextant observations of July 5th, 10th, and 15th, I have computed the following elements for the Comet. They can be considered only as approximate, as my sight is failing, and the intervals rather shorter than I could wish:—

Perihelion long.	249° 38′
,, dist.	·825
,, passage	June 11^{d} 20^{h}·7
Long. ascending Node	99° 1′
Inclination	85° 33′

Motion direct.

"This differs considerably from Hind's May elements, as given in my former letter; but with an orbit almost perpendicular to the ecliptic, such differences are to be expected.

"From these elements I find that the Comet on the evening of July 5th was only eleven millions of miles from the Earth. Probably (for I have not calculated it) it was not much more than five millions, about the beginning of the month. This will account satisfactorily for its immense tail, and great velocity. I should be very thankful for any observations before the 5th and after the 15th. A week or two more will bring the determination from Europe of what the elements really are.

"Hind gives a group of four comets, with large inclinations, direct motions, and perihelion distances varying from ·7 to ·8, ·9, and 1, for which periods have been computed of about 75 years. Another will now be added to the group; for I have no doubt that this comet is identical with the 1st of 1781, of which the elements were found by Mechain.

"They are as follows:

Perihelion long.	239°	Hind has 243
,, dist.	·755	
,, passage	July 7^{d}	
Ascending Node	83°	
Inclination	81°	
	Motion direct.	

If so, a comet with a known period of 80 years will take rank with Halley's. If they could be found, it would be interesting to recompute Mechain's calculations.

"On looking through Hind's list, I find a Comet in 1797 not very unlike, and another in 68 B.C., which I feel strongly disposed to identify with our Comet. From the obscure notices of the period in the Chinese annals, it appears to have had a perihelion distance of 8, and an inclination of 70°. Adding 68 to 1861, we have 1929, a near enough multiple of 80, when it is considered how much the cometary periods vary.

"No doubt all this has been discovered, and much more satisfactorily, in Europe; but it is new at least in this country.

"W. S. M."

"*Chinsurah, 19th July*, 1861.

"My observations were,—

		R.A.	N. Decl.
	h m	° ′ ″	° ′ ″
July 5	3 12	172 31 30	66 40 20
10	3 12	206 10 15	59 41 17
15	3 12	215 41 0	54 50 32

The time is Greenwich mean time."

Extract of a Letter from Mr. Lassell.

(*Communicated by Prof. De Morgan.*)

"The principal work, however, I have hitherto done is the perfectly successful erection of the four-foot Equatoreal, on a most favourable site near Fort Tigné, on the north side of the Quarantine Harbour in this island. It differs a little from the lithographic drawing in the possession of the Society, as it is now erected in the latitude for which it was constructed, and not, as in that sketch, with the polar axis inserted in an excavation, to allow of its temporary use in the latitude of Liver-

pool. It is, indeed, much more substantially erected, with a view to its permanent efficiency in this far more favourable climate, where I trust it will do all that can reasonably be expected from 48 inches of aperture. I am glad to say that I have had no accident whatever in the erection, nor in the transmission of the mirrors, that which is now in the tube, after the dissolution and removal of its coat of varnish, having come out as fine in lustre as if it had been polished only yesterday.

"I have hitherto only cursorily examined a few objects, some of the minor conveniences of the apparatus not being yet quite completed; but I see enough to convince me that I have now a much more powerful and efficient telescope than I have ever possessed before. We are now passing through the most unsettled portion of the year, and I have found this so-called winter less favourable for the use of a large telescope than the same portion of the year 1852; I can, therefore, scarcely tell what the telescope will do under the most favourable circumstances. In turning it, however, upon the Nebula in *Orion*, I see so much more of its wonderful constitution, that I feel very ill-satisfied with my former drawings, and must begin anew. I was, indeed, at first rather surprised, perhaps disappointed, not to see many more new stars; but, on reflection, I am inclined to think this may be satisfactorily accounted for by the great increase of brightness of the Nebula rendering the eye less sensible to very minute points of light. And this view is confirmed by observations of stars in the day-time with the transit instrument, which, I suppose from the greater light of the sky here, are scarcely so visible as in England.

"The fainter and interior satellites of *Uranus*, *Umbriel* and *Ariel*, though as much inferior in brightness to what have been called the bright satellites, as *Rhea*, or even *Dione*, among the satellites of *Saturn*, is to *Titan*, are conspicuous objects; and, with a magnifying power of 1060, not to be overlooked on any tolerable night. If *Uranus* has any other satellites than these, they must be very much smaller or fainter.

"The only indication of *Saturn's* ring at present visible is its shadow as a thin, hard, dark, rough line across or along the equator of the planet, which is now so full of sharp belts over almost all its surface, that it might be taken for a view of *Jupiter* with a smaller telescope. On the 25th instant all the satellites but *Japetus* were in the field of view at once; *Mimas* and *Enceladus* within a very few seconds, indeed, of the limb of the planet; the latter almost touching it. Another of the nearer satellites emerged from behind the preceding limb during observation.

"P. S.—It is, perhaps, scarce worth mentioning, and yet it struck me as either a fortunate coincidence or an evidence of care, that I have had no occasion to touch the adjustments of either Right Ascension or Polar Distance since the telescope was

first directed to a celestial object; and, without having made any observations expressly for the purpose, I have not been made aware of any obvious error, in turning the telescope upon such objects as I have examined.

"9 *Piazza Slierna, Malta, 28th January*, 1862."

And, in a letter dated the 30th January, Mr. Lassell writes,—"The detention of the mail allows me to add a postscript to my letter of yesterday, to inform you of my having discovered, last night, a new star within the trapezium in the great Nebula of *Orion*. It is situated near *Theta*, the principal star; appears to be about a full magnitude less than that known as 'the sixth star;' and is about one sixteenth of its distance from *Theta*. Its angle of position from *Theta* is about 100° less than that of 'the sixth star;' and, consequently, points a little eastward of the star at the opposite angle of the trapezium. For verification I annex a diagram.

"I suspect that the position-angle of 'the sixth star' has considerably increased since 1852."

Results of Meridional Observations of Small Planets; Occultation of a Star by the Moon; and Phenomena of Jupiter's Satellites; observed at the Royal Observatory, Greenwich, during the months of January and February, 1862.

(*Communicated by the Astronomer Royal.*)

Metis ⑨.

Mean Solar Time of Observation.	R. A. from Observation.	N.P.D. from Observation.
	h m s	h m s	° ′ ″
1862, Jan. 16	13 25 54·4	9 10 46·25	64 58 1·11
22	12 56 31·4	9 4 57·79	64 17 28·31
24	12 46 36·6	9 2 54·52	64 4 35·41
25	12 41 37·8	9 1 51·44	63 58 19·81
27	12 31 39·3	8 59 44·39	63 46 12·19
28	12 26 39·7	8 58 40·53	63 40 17·65
Feb. 3	11 56 40·3	8 52 15·56	63 8 28·38
7	11 36 48·6	8 48 6·80	62 50 58·54
10	11 22 2·0	8 45 7·42	62 39 54·42

Eunomia (15).

Mean Solar Time of Observation.	R.A. from Observation.	N.P.D. from Observation.
h m s	h m s	° ′ ″
1862, Jan. 25 12 16 36·2	8 36 45·72	75 52 24·27
27 12 6 36·0	8 34 36·92	75 53 2·04
Feb. 10 10 57 31·8	8 20 33·16	75 55 8·66
18 10 19 32·5	8 14 0·05	75 56 19·34

Thalia (23).

Mean Solar Time of Observation.	R.A. from Observation.	N.P.D. from Observation.
h m s	h m s	° ′ ″
1862, Feb. 18 12 53 43·6	10 48 36·56	61 32 23·96

Calypso (53).

Mean Solar Time of Observation.	R.A. from Observation.	N.P.D. from Observation.
h m s	h m s	° ′ ″
1862, Feb. 10 12 42 14·8	10 5 33·35	77 55 8·14

All the observations of N.P.D. have been corrected for refraction and parallax.

The disappearance of *d Geminorum*, on 1862, Jan. 14, at the Moon's dark limb was observed by Mr. Carpenter to take place at $6^h\ 29^m\ 58^s{\cdot}9$ G.M.T. The star was extremely faint.

Phenomena of Jupiter's Satellites.

Day of Observation. 1862.	Satellite.	Phenomenon.	Mean Solar Time. h m s	Observer.
Jan. 24	IV	Occ. reapp. last cont.	14 35 0·4	C.
28	I	Eclipse, disapp.	12 52 48·0	E.
Feb. 18	IV	Egress, first cont.	11 12 36·9	S.
,,	IV	,, last cont.	11 17 17·2	S.
,,	III	Occ. reapp. first cont.	11 53 45·2	S.
,,	III	,, last cont.	12 2 6·9	S.

Feb. 18, The image of the planet exceedingly tremulous, and sometimes faint from cloud.

The initials S., E., C., are respectively those of Mr. Stone, Mr. Ellis, and Mr. Criswick.

Occultation observed at Highbury. By T. W. Burr, Esq.

1862, January 4th. The star κ *Aquarii* disappeared at the dark limb of the Moon, at $1^h\ 34^m\ 49^s{\cdot}6$ local sidereal time, $= 6^h\ 38^m\ 47^s{\cdot}1$, Greenwich mean time. This occultation was instantaneous, but there had been a disappearance and reappearance about a minute previously, of an abnormal character. The Moon's dark limb was visible, and approaching the star extinguished it at $1^h\ 33^m\ 12^s{\cdot}6$ L.S.T. After watching some little while, to be sure of the fact, I verified my time and recorded the observation, but on returning to the telescope was surprised to see the star again visible close to the Moon's edge, and finally disappearing as above mentioned. This is the only occasion out of the many occultations I have witnessed when anything unusual has occurred, but the observation in this case may be relied on.

The ultimate reappearance was not seen, owing to a passing cloud.

The time was checked by transits of α *Andromedæ* and β *Ceti*, corrected by Mr. Epps' tables (*Mem. Ast. Soc.* vol. iv.). Telescope, focal length, 4 ft. 4 in. Aperture, $3\frac{3}{8}$ in. Power, 173. Longitude, 24^s W.

March 10, 1862.

Solar Eclipse of 31st December, 1861, observed at Goree, and at St. Louis, Senegal, by Officers of the French Imperial Navy.

By authority of the Duke of Newcastle, Secretary of State for the Colonies, Mr. Hind communicated to the Society the original of the Report of these observations, taken by the direction of the Governor of the French colony of Senegal; Mr. Hind states that the observations are due to the warm interest evinced by Capt. Washington, the Hydrographer of the Admiralty, on the occasion: he having forwarded a few instructions drawn up by the Astronomer Royal to the Governor of the Gambia, with a request that he would procure from the authorities at Goree a report of the eclipse.

The observers at St. Louis were MM. Vallon, Ribel, and De Marteville; those at Goree, MM. Poulain and Dutaillis. The eclipse commenced at St. Louis about $0^h\ 4^m$, and the last contact was observed at $2^h\ 57^m\ 36^s{\cdot}1$. At Goree the Moon had entered about one-tenth of the diameter on the solar disk at $0^h\ 3^m\ 35^s$, and the eclipse terminated at $2^h\ 51^m\ 18^s$.* For

* The times throughout are mean solar time of St. Louis or of Goree, according to the place of observation.— Ed.

want of a telescope of sufficient power the red prominences were not observed at Goree. The following are extracts from the Report: —

"*Saint Louis.*— A $1^h\ 30^m\ 50^s$ pendant que le croissant du Soleil passe rapidement à l'occident de la Lune, une magnifique protubérance se détache du coté opposé à 12° environ au-dessus du diamètre horizontal du Soleil. D'abord voisine de la corne supérieure orientale, sa base en partie cachée par la Lune embrasse 5 ou 6 dégrés du disque solaire; le sommet, éclairé d'un beau rose, se dessine franchement sur une portion de couronne d'aspect rayonnant, jetant quelques ombres très mobiles qui descendent du nord au sud comme le feraient les ailes rapides d'un moulin à vent. Ces rayons d'ombre convergent au centre du Soleil.

"Ce beau phénomène dure de $1^h\ 30^m{\cdot}50$ à $1^h\ 34^m\ 44^s$, heure à laquelle la protubérance et tout jeu de lumière s'effacent derrière le disque de la Lune. Cette protubérance étant très voisine de la perpendiculaire à la corde du croissant solaire, son immersion a été suivie avec soin. La base à d'abord disparu, puis graduellement toutes ses parties, dans le tems du mouvement de l'éclipse; enfin le sommet, au moment ou l'épaisseur du croissant était mesurée au cercle $1'\ 10''$.

"Il est évident que cette mesure correspond à peu près exactement à la hauteur du sommet de la protubérance sur le disque du Soleil, auquel elle est *sans nul doute* attachée, ces deux éléments étant situés aux extrémités d'un même diamètre.

"Nous croyons que la protubérance est l'effet d'un amas de nuages solaires; le telescope permettait d'en compter les dentelures au nombre de trois principales sur son versant supérieur. La partie inférieure moins étendue paraissait en surplomb sur le bord de la Lune. Le sommet, mieux éclairé que la base, dont la Lune masquait le lieu de jonction au disque solaire, atteint son maximum d'éclat, une minute avant son immersion.

"*Gorée.* — Faute de téléscope, on ne peut que reconnaître le beau phénomène de la couronne qui environne les disques superposés au point de produire l'effet d'une éclipse annulaire.

"*Saint Louis.* — La portion de couronne visible à l'orient de la Lune embrassait un secteur de 60 à 70 dégrés. D'un beau rose près des disques des astres, les rayons allaient s'affaiblissant dans la teinte sombre du ciel. Sa structure était à rayons, et elle a paru suivre l'immersion graduelle de la protubérance.

"La lumière de cette portion de couronne a été observée à éclat; effet produit par les ombres rapides que nous avons signalées plus haut.

"*Gorée.*—Vers $1^h 25^m$ on voit des franges noires se dessiner sur le mur de la terrasse d'observation: à $1^h\ 26^m$ l'éclipse paraît centrale; une cercle lumineux existe autour du disque lunaire; son épaisseur est de $\frac{1}{3}$ de doigt; des aigrettes lumineuses émanent de tous cotés; à droite et à gauche elles sont

parfaitement symétriques ; au-dessus, il y en a une plus longue et plus large ; au-dessous elles sont plus courtes.

" Il règne dans les environs une obscurité qu'on peut comparer à la lueur d'un crépuscule dans les plus belles régions de l'Europe. *Vénus* et trois autres étoiles paraissent au firmament à $2^h\ 34^m\ 30^s$; la distance de *Vénus* au Soleil a été observée 46° 30′.

" *Saint Louis.*— Trois groupes de taches ont paru dans le 2e quartier du Soleil, deux autres plus petits à la partie inférieure du 4e quartier.

" Ces taches ont en général un noyau brun dont les bords sont *tranchés* sur l'enveloppe vaporiforme qui la circonscrit. Cette enveloppe, d'une teinte neutre affaiblie, a un contour bien accusé sur le disque du Soleil.

" Nous affirmons que pendant la durée de l'observation aucune de ces taches n'a changé de position relative, de forme absolue, ni de teinte.

" Doivent-elles leur existence au déchirement de la photosphère qui laisserait apercevoir le noyau central du Soleil, ou son enveloppe de nuages obscurs, suivant Arago ? Cette explication satisfait pleinement la vue. Sont-elles le résultat d'une accumulation de nuages solaires de même nature que les protubérances? Cette dernière supposition fort ingénieuse ne nous paraît cependant pas soutenable, et n'explique ni la figure étrange des taches, ni leur immobilité pendant 3 heures, ni enfin la permanence des effets de l'éclairage de toutes leurs parties.

" Le centre des taches principales renfermait des espaces lumineux disposés irrégulièrement. La Lune, dans son ascension, a successivement recouvert ces groupes ; ils devenaient d'autant plus visibles que son disque s'en approchait d'avantage. Aucune de ces taches en s'immergeant, ou en reparaissent, n'a subi la moindre altération de forme. Les cornes du Soleil se sont effilées sans distorsion apparente.

" Les baromètres au moment de la plus grande obscurité sont descendus de 763·5 à 762·5.

" Les thermomètres au Soleil de 34° centigrades à 23°·6.

" Les thermomètres à l'ombre de 28°·3 centigrades à 23°·5.

" L'arguille aimantée n'éprouve aucune oscillation.

" Les deux hygromètres accusent 10 centièmes d'humidité en plus.

" Enfin, la face du Soleil a paru très homogène, à part quelques groupes irréguliers de facules lumineuses qui environnent les deux petits groupes de taches inférieures.

" *Gorée.*—Le thermomètre à l'ombre est descendu de 25°·8 à 20 dégrés. La courbe de la chaleur parait d'accord avec celle de la lumière.

" Le baromètre a varié entre 762 et 763.

" Le psycromètre, formé de deux thermomètres, un sec et un mouillé, donnait en centièmes, au moyen des tables, la

quantité d'humidité renfermée dans l'air. Il a accusé 12 centièmes d'augmentation.

"Le phénomène des franges noires a été observé un instant avant l'occultation; elles avaient, sur une surface verticale dirigée sensiblement de l'ouest à l'est, une direction de 45° par-dessus l'épaule gauche; leur largeur et leur espacement étaient à-peu-près égaux et mesuraient environ 10 centimètres. La nuance était une ombre claire qui a paru dégradée sur les bords.

"Avant et après l'occultation, les animaux domestiques et les plantes ont été influencés comme à l'approche de la nuit et au lever du jour.

"Il y avait à Gorée un raz de marée exceptionnel.

"*Saint Louis.*—*Vénus* a paru dans le S. E. au moment de la plus grande obscurité. Les édifices blanchis à la chaux paraissaient gris jaunâtres. Le rouge conservait sa couleur. La moindre épaisseur du croissant solaire a été mesurée 57″ à $1^h\ 30^m\ 49^s$."

Sir Thomas Maclear has communicated Observations of the Right Ascensions and North Polar Distances of Comet I., 1861, made at the Royal Observatory, Cape of Good Hope, from 18th August to 6th September, 1861; and in a letter, dated 21st December, 1861, addressed to the Astronomer Royal, he writes:—

"Dr. Pape's ephemeris reached the Cape too late. By extending it from August 6th, its limit, the comet was readily found—then too faint, however, for precise work.

"I detected Encke's Comet on the 4th instant, but the light was too feeble for place, and a few days after the moonlight extinguished it altogether. It is now workable, and I hope for a good series of observations.

"It would be well to get up a rough working ephemeris of all the periodic comets several months beforehand, and to circulate them freely. By rough I mean with omission of the perturbation corrections."

The observations will probably be printed, along with those of Comet II., 1860, in the *Memoirs* of the Society.—Ed.

A paper "On Astronomical Refraction," by Mr. Andrew Yeates, was read at the Meeting.

Among the Presents to the Society, there were presented by Mr. H. G. Bohn of York Street, Covent Garden :—

Opusculum Johannis de Sacrobusto Sphericum. Lipsiæ, 4to. *s.a.*

Ditto. Liptczk, 4to. 1499; with additional matter in manuscript.

Flores Albumasaris, Ehrhardt Ratdolt. 4to. 1495. At the end there are about 22 pages of Astrological matter, beautifully written in a monastic or court hand.

Discovery of a Companion of Sirius.

In the *Ast. Nach.*, No. 1353, Prof. Bond communicates the discovery of a Companion of *Sirius*, made on the evening of Jan. 31 by Mr. Clark, with his new object-glass of 18½ inches aperture. Prof. Bond was able to observe it with the Refractor of 15 inches, at the Observatory of Harvard College, as follows:

1862, Feb. 10, Angle of Position	85° 15	±1° 1
Distance ...	10″·37	±0″·2

when the images were tranquil, the Companion was seen distinctly enough, but on account of the atmospheric disturbances these moments are quite rare.

The Companion was seen at Paris by M. Chacornac, the 20th March, with the telescope with silvered mirror of 80 centimeters, constructed according to the plans and under the direction of M. Foucault.

It appears from the *Cosmos* of 28th March, that Dr. Peters does not accept the identity of the Companion thus discovered with that which he had calculated.

Discovery and Elements of a New Minor Planet, (72).

In the *Ast. Nach.*, No. 1353, Prof. Bond communicates also the discovery of this new Minor Planet, under the following circumstances. Mr. Safford having had occasion to refer to the positions of *Maia* (66), obtained by Dr. Peters at Hamilton

College, and published in Brünnow's *Astronomical Notices*, No. 20, was surprised to find that only three of the series, namely, the places for 1861, May 9, 11, and 12, could be reconciled with the Cambridge (U.S.) observations. A reference to Mr. Hall's Ephemeris of *Maia* (*Ast. Nach.*, No. 1315), showed that the Ephemeris represented the Cambridge observations of *Maia* from April 9 to May 27, nine in number, and also the first three of those of Dr. Peters, but that it did not represent the later observations of Dr. Peters. It was at once conjectured that in the interval between May 12 and May 29, when clouds and moonlight intervened to prevent a close following of *Maia*, which was only of the 13th mag., Dr. Peters had lost its trace, and on resuming his observations had fallen on a new Planet. Mr. Safford, after obtaining corrected positions of the comparison stars of Dr. Peters, obtained the following elements,—

1861, May 29·375 Washington M.T.

L	= 213° 3′ 24″·1	Mean Equinox 1861·0
ϖ	= 329 22 16·5	
Ω	= 208 1 28·0	
i	= 5 23 16·2	
φ	= 6 50 26·0	
μ	= 1129″·372	
log a	= 0·331446	

It is noticed that (72) has the least mean distance yet recognised among the Minor Planets.

RECENT PUBLICATION.

A Handbook of Descriptive and Practical Astronomy. By Geo. F. Chambers, F.R.G.S. 8vo. London, 1861.

The author remarks that his book is intended to occupy a middle position between purely elementary books on the one hand, and advanced treatises on the other hand; to be attractive to the general reader, useful to the amateur, and "handy" also as an occasional book of reference to the professional astronomer. Pains have been taken to incorporate the most recent discoveries in all branches of the science, and the Work appears to be very complete in its range of subjects; and in general to be fully entitled to take up the position claimed for it by the author. The chronological tables of Astronomers and of Discoveries in Book 9, and the tables of Eclipses, Comets, Star Catalogues, &c., contained in the Appendices, will probably be found useful for reference.

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MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. XXII. *Supplemental Notice.* No. 5 *bis.*

*On the Secular Acceleration of the Moon's Mean Motion.**

By A. Cayley, Esq.

The present Memoir exhibits a new method of taking account, in the Lunar Theory, of the Variation of the Excentricity of the Sun's Orbit. The approximation is carried to the same extent as in Prof. Adams' Memoir "On the Secular Variation of the Moon's Mean Motion" (*Phil. Trans.*, vol. cxliii. (1853), pp. 397–406); and I obtain results agreeing precisely with his, viz., besides his new periodic terms in the longitude and radius vector, I obtain in the longitude the secular term

$$\left(-\frac{3}{2}m^2 + \frac{3771}{64}m^4\right) \int \left(e'^2 - \mathrm{E}'^2\right) n\, dt,$$

and in the quotient radius, or radius vector divided by the mean distance, the secular term

$$\left(\frac{3}{4}m^2 - \frac{1973}{64}m^4\right) \left(e'^2 - \mathrm{E}'^2\right),$$

which is, in fact, as will be shown, included implicitly in the results given in Professor Adams' Memoir. In quoting the foregoing results, I have written $e'^2 - \mathrm{E}'^2$ in the place of $(e' + f' t)^2 - e'^2 = 2 e' f' t$, which in the notation of the present Memoir it should have been; and I purposely refrain from here explaining the precise signification of the symbols: this is carefully done in the sequel. The method appears to me a very simple one in principle; and it possesses the advantage that it is not incorporated step by step with a lunar theory in which the

* This Memoir, an abstract of which appeared in the December number, p. 32, has been ordered by the Council to be printed *in extenso*.—ED.

excentricity of the Sun's orbit is treated as constant; but it is added on to such a lunar theory, giving in the Moon's coordinates the supplementary terms which arise from the variation of the solar excentricity, and thus serving as a verification of any process employed for taking account of such variation.

I have given the details of the work in a series of Annexes, 1 to 23: this appears to me the best course for presenting the investigation in a readable form.

I.

The inclination and excentricity of the Moon's orbit, and, *à fortiori*, the variation of the position of the Ecliptic, and the Sun's latitude, are neglected; and the longitudes are measured from a fixed point in the Ecliptic. I write

n, the actual mean motion of the Moon at a given epoch;

viz., it is assumed that the mean longitude at the time t is $\varepsilon + n\,t + n_2\,t^2 +$ &c. where ε, n, n_2, &c. are absolute constants; and, moreover,

a, the calculated mean distance of the Moon;

that is, $n^2\,a^3$ is the sum of the masses of the Earth and Moon; a is therefore an absolute constant; and, in like manner,

n', the actual mean motion of the Sun at the same epoch,

a', the calculated mean distance of the Sun;

that is, if it were necessary to pay attention to the secular variation of the mean motion of the Sun, the assumption would be that the mean longitude was $\varepsilon' + n'\,t + n'_2\,t^2 +$ &c., ε', n', n'_2, &c. being absolute constants, and $n'^2\,a'^3$ the sum of the masses of the Sun and Earth; a' would thus also be an absolute constant. But for the purpose of the present investigation the secular variation of the mean longitude of the Sun is neglected, or it is assumed that the mean longitude of the Sun is $\varepsilon' + n'\,t$, ε', n' being absolute constants; and that $n'^2\,a'^3$ is the sum of the masses of the Sun and Earth, a' being thus also an absolute constant.

I put also

m, the ratio of the mean motions of the Sun and Moon;

that is,

$$m = \frac{n'}{n}, \text{ or } n' = m\,n;$$

m is also an absolute constant.

The Sun is considered as moving in an elliptic orbit, the excentricity whereof is $e' + \delta e'$ or $e' + f' t$, e' and f' being absolute constants; the longitude of the Sun's perigee may be taken to be $\varpi' + (1 - c') n' t$; so that the mean anomaly g' is $= n' t + \varepsilon - [\varpi' - (1 - c') n' t] = c' n' t + \varepsilon' - \varpi'$; c', ϖ', being absolute constants; but c' is in fact treated as being $= 1$. Hence, if r', v' are the radius vector and longitude of the Sun, we have

$$r' = a' \operatorname{elqr} (e' + \delta e', g')$$

$$v' = \varpi' + (1 - c') n' t + \operatorname{elta} (e' + \delta e', g')$$

$$= n' t + \varepsilon + [\operatorname{elta} (e' + \delta e', g') - g'],$$

where

$$g' = c' n' t + \varepsilon' - \varpi'.$$

In the expression for the disturbing function the Sun's mass is taken to be $= n'^2 a'^3$, or, what is the same thing, $= m^2 n^2 a'^3$.

Let r, v be the radius vector and longitude of the Moon; then, taking the usual approximate expression of the Disturbing Function, the equations of motion are

$$\frac{d}{dt}\frac{dr}{dt} - r\left(\frac{dv}{dt}\right)^2 + \frac{n^2 a^3}{r^2} = m^2 n^2 a'^3 \frac{r}{r'^3}\left(\frac{1}{2} + \frac{3}{2}\cos 2v - 2v'\right),$$

$$\frac{d}{dt}\frac{r^2 dv}{dt} = m^2 n^2 a'^3 \frac{r^2}{r'^3}\left(-\frac{3}{2}\sin 2v - 2v'\right);$$

It will be convenient to assume

ϱ, the quotient radius of the Moon's orbit,

ϱ', the quotient radius of the Sun's orbit;

that is

$$r = \varrho a, \quad r' = \varrho' a'.$$

The equations of motion thus become

$$\frac{d}{dt}\frac{\delta\varrho}{dt} - \varrho\left(\frac{dv}{dt}\right)^2 + \frac{n^2}{\varrho^3} = m^2 n^2 \mathrm{P},$$

$$\frac{d}{dt}\left(\varrho^2 \frac{dv}{dt}\right) = m^2 n^2 \mathrm{Q},$$

where for shortness

$$\mathrm{P} = \frac{\varrho}{\varrho'^3}\left(\frac{1}{2} + \frac{3}{2}\cos 2v - 2v'\right),$$

$$\mathrm{Q} = \frac{\varrho^2}{\varrho'^3}\left(-\frac{3}{2}\sin 2v - 2v'\right),$$

in which

$$\varrho' = \text{elqr}\,(e' + \delta e', g')$$
$$v' = n' t + \varepsilon' + [\text{elta}\,(e', g') - g']$$

I now change the notation by writing $\varrho' + \delta\varrho'$, $v' + \delta v'$, in the place of ϱ', v', respectively, using henceforward ϱ', v' to denote

$$\varrho' = \text{elqr}\,(e', g')$$
$$v' = n' t + \varepsilon' + [\text{elta}\,(e, g') - g'];$$

and I write also $\varrho + \delta\varrho$, $v + \delta v$, in the place of ϱ, v, using henceforward ϱ, v to denote the solutions of the equations obtained from the equations of motion by writing therein ϱ', v' instead of the complete values $\varrho' + \delta\varrho'$, $v' + \delta v'$.

Suppose, in like manner, that the complete values of P, Q are denoted by $\mathrm{P} + \delta\mathrm{P}$, $\mathrm{Q} + \delta\mathrm{Q}$, where

$$\delta\mathrm{P} = \frac{d\mathrm{P}}{d\varrho}\delta\varrho + \frac{d\mathrm{P}}{dv}\delta v + \frac{d\mathrm{P}}{d\varrho'}\delta\varrho' + \frac{d\mathrm{P}}{dv'}\delta v',$$

with a like value for $\delta\mathrm{Q}$, the first powers of $\delta\varrho$, δv, $\delta\varrho'$, $\delta v'$ being alone attended to. Then, observing that the equations of motion are satisfied when $\delta\varrho$, δv, $\delta\varrho'$, $\delta v'$ are neglected, we have, it is clear,

$$\frac{d}{dt}\left(\frac{d\delta\varrho}{dt} - \delta\varrho\left(\frac{dv}{dt}\right)^2 - 2\varrho\frac{dv}{dt}\frac{d\delta v}{dt} - \frac{2n^2}{\varrho^3}\delta\varrho\right) = m^2 n^2 \delta\mathrm{P},$$

$$\frac{d}{dt}\left(\varrho^2\frac{d\delta v}{dt} + 2\varrho\delta\varrho\frac{dv}{dt}\right) = m^2 n^2 \delta\mathrm{Q}.$$

The second of these equations gives

$$\varrho^2\frac{d\delta v}{dt} + 2\varrho\delta\varrho\frac{dv}{dt} = m^2 n^2 (\mathrm{C} + \int d\mathrm{Q}\, dt),$$

where the constant of integration, C, is to be so determined that δv may not contain any term of the form kt (for any such term is taken to be included in the term nt of $v + \delta v$). Multiplying the equation just obtained by $\frac{2}{\varrho}\frac{dv}{dt}$, and adding it to the first equation, we have

$$\frac{d^2\delta\varrho}{dt^2} + \left\{3\left(\frac{dv}{dt}\right)^2 - \frac{2n^2}{\varrho^3}\right\}\delta\varrho = m^2 n^2\left(\delta\mathrm{P} + \frac{2}{\varrho}\frac{dv}{dt}(\mathrm{C} + \int d\mathrm{Q}\, dt)\right),$$

which, with the above-mentioned integral equation, are the

equations for the solution of the problem; but it will be convenient to write them under the slightly different form

$$\frac{d^2\delta\varrho}{dt^2}+n^2\delta\varrho=\left\{n^2+\frac{2n^2}{\varrho^3}-3\left(\frac{dv}{dt}\right)^2\right\}\delta\varrho+m^2n^2\left\{\delta P+\frac{2}{\varrho}\frac{dv}{dt}\left(C+\int dQ\,dt\right)\right\},$$

$$\frac{d\delta v}{dt}=-\frac{2}{\varrho}\frac{dv}{dt}\delta\varrho+\frac{m^2n^2}{\varrho^2}\left(C+\int\delta Q\,dt\right).$$

In these equations C is determined, as above, by the condition that $\frac{d\delta v}{dt}$ may contain no constant term; the values of ϱ', v', $\delta\varrho'$, $\delta v'$ are of course given by the theory of elliptic motion, and those of ϱ, v are given by the ordinary lunar theory, in which the excentricity of the solar orbit is treated as a constant; and then, $\delta\varrho$, δv being obtained by integrating the equations, the radius vector and longitude of the Moon are $a(\varrho+\delta\varrho)$ and $v+\delta v$ respectively.

We have

$$P=\frac{\varrho}{\varrho'^3}\left(\frac{1}{2}+\frac{3}{2}\cos 2v-2v'\right),$$

$$Q=\frac{\varrho}{\varrho'^3}\left(-\frac{3}{2}\sin 2v-2v'\right).$$

Moreover, by the lunar theory, observing that Plana's a is or may be considered identical with the a of the present Memoir, and putting also

$$\tau=nt+\varepsilon-(n't+\varepsilon'),$$

we have

$$\begin{aligned}\frac{1}{\varrho}=1&+\frac{1}{6}m^2-\frac{3}{4}m^2e'^2\\&-\frac{3}{2}m^2e' && \cos g'\\&+m^2-\frac{5}{2}m^2e'^2 && ,,\ 2\tau\\&+\frac{7}{2}m^2e' && ,,\ 2\tau-g'\\&-\frac{1}{2}m^2e' && ,,\ 2\tau+g'\\&-\frac{9}{4}m^2e'^2 && ,,\ 2g'\\&+\frac{17}{2}m^2e'^2 && ,,\ 2\tau-2g'\\&\circ\,m^2e'^2 && ,,\ 2\tau+2g',\end{aligned}$$

$$\begin{aligned} v = {} & n t + \varepsilon \\ & - 3\, m e' + \circ\, m^2 e' && \sin g' \\ & + \frac{11}{8} m^2 - \frac{55}{16} m^2 e'^2 && ,, \ 2\tau \\ & + \frac{77}{16} m^2 e' && ,, \ 2\tau - g' \\ & - \frac{11}{16} m^2 e' && ,, \ 2\tau + g' \\ & - \frac{9}{4} m e'^2 + \circ\, m^2 e'^2 && ,, \ 2 g' \\ & + \frac{187}{16} m^2 e'^2 && ,, \ 2\tau - 2 g' \\ & \circ\, m^2 e'^2 && ,, \ 2\tau + 2 g', \end{aligned}$$

where the series are carried as far as m^2 and e'^2; the terms in e'^2 are given, as I shall have occasion to refer to them, but they are not used in the investigation, and, omitting them, the values are

$$\begin{aligned} \frac{1}{\varrho} = {} & 1 + \frac{1}{6} m^2 \\ & - \frac{3}{2} m^2 e' && \cos g' \\ & + \ m^2 && ,, \ 2\tau \\ & + \frac{7}{2} m^2 e' && ,, \ 2\tau - g' \\ & - \frac{1}{2} m^2 e' && ,, \ 2\tau + g', \end{aligned}$$

$$\begin{aligned} v = {} & n t + \varepsilon \\ & - 3\, m e' && \sin g' \\ & + \frac{11}{8} m^2 && ,, \ 2\tau \\ & + \frac{77}{16} m^2 e' && ,, \ 2\tau - g' \\ & - \frac{11}{8} m^2 e' && ,, \ 2\tau + g' \end{aligned}$$

$$(g' = c' m n t + \text{const.}, \quad 2\tau = (2 - 2m) n t + \text{const.})$$

For the co-ordinates of the Sun we have

$$\begin{aligned}\frac{1}{\varrho'} &= 1 \\ &\quad + e' \cos g' \\ &\quad + e'^2 \;,, \; 2g', \\ v' &= n't + \varepsilon' \\ &\quad + 2e' \sin g' \\ &\quad + \frac{5}{4} e'^2 \;,, \; 2g',\end{aligned}$$

the series being carried as far as e'^2; but the terms in e'^2 are only used for the formation of $\delta\varrho'$, $\delta v'$; and, omitting them, we have

$$\begin{aligned}\frac{1}{\varrho'} &= 1 \\ &\quad + e' \cos g', \\ v' &= n't + \varepsilon' \\ &\quad - 2e' \sin g'.\end{aligned}$$

If $e' + f't$ is written for e', then the value of $\delta e'$ is $= f't$; but as only the terms multiplied by the simple power f' are attended to, we may for convenience write $\delta e' = t$, the factor f' being restored in the final results: we thus have

$$\begin{aligned}\delta\frac{1}{\varrho'} &= 1 \quad t\cos g' \\ &\quad + 2e' \;,, \; 2g', \\ \delta v' &= 2 \quad t\sin g' \\ &\quad + \frac{5}{2} e' \;,, \; 2g',\end{aligned}$$

and we may add the equations

$$\begin{aligned}\frac{dv}{dt} &= \overbrace{1}^{n\times} \\ &\quad - 3m^2 e' \cos g' \\ &\quad + \frac{11}{4} m^2 \;,, \; 2\tau \\ &\quad + \frac{77}{8} m^2 e' \;,, \; 2\tau - g' \\ &\quad - \frac{11}{8} m^2 e' \;,, \; 2\tau + g',\end{aligned}$$

$$\delta \frac{1}{\varrho'^3} = 3\, e'\ t$$
$$+ 3\ \ t \cos g'$$
$$+ \frac{9}{2} e'\ \ ,,\ 2g',$$
$$\frac{\delta v'}{\varrho'^3} = 2\ \ t \sin g'$$
$$+ \frac{11}{2} e'\ \ ,,\ 2g',$$

which will be found useful.

II.

Proceeding now to the development of the solution, we have

$$\delta P = \frac{1}{\varrho'^3}\left(\frac{1}{2} + \frac{3}{2}\cos 2v - 2v'\right)\delta\varrho$$
$$+ \frac{\varrho}{\varrho'^3}\left(-\frac{3}{2}\sin 2v - 2v'\right)\delta v$$
$$+ \varrho\left[\left(\frac{1}{2} + \frac{3}{2}\cos 2v - 2v'\right)\delta\frac{1}{\varrho'^3} + \left(3\sin 2v - 2v'\right)\frac{\delta v'}{\varrho'^3}\right],$$

where the terms containing $\delta\varrho$ and δv are (see Annex 1)

$\frac{1}{2}$	
$+\frac{3}{2} e' \cos g'$	
$+\frac{3}{2}$,, 2τ	$\delta\varrho$
$+\frac{21}{4} e'$,, $2\tau - g'$	
$-\frac{3}{4} e'$,, $2\tau + g'$	

$-3 \sin 2\tau$	
$-\frac{21}{2} e'$,, $2\tau - g'$	$\delta v,$
$+\frac{3}{2} e'$,, $2\tau + g'$	

and also

$$\delta Q = \frac{\varrho}{\varrho'^3}(-3\sin 2v - 2v')\,\delta\varrho + \frac{\varrho^2}{\varrho'^3}(-3\cos 2v - 2v')\,\delta v + \varrho^2\left[\left(-\frac{3}{2}\sin 2v - 2v'\right)\delta\frac{1}{\varrho'^3} + 3\cos 2v - 2v')\,\frac{\delta v'}{\varrho'^3}\right],$$

where the terms containing $\delta\varrho$ and δv are (see Annex 2)

$-3 \quad \sin 2\tau$	$\delta\varrho$
$-\frac{21}{2}e' \;,,\; 2\tau - g'$	
$+\frac{3}{2}e' \;,,\; 2\tau + g'$	

$-3 \quad \cos 2\tau$	$\delta v,$
$-\frac{21}{2}e' \;,,\; 2\tau - g'$	
$+\frac{3}{2}e' \;,,\; 2\tau + g'$	

but the additional term $\frac{171}{4}m^2 e' \cos g'$ is ultimately added (see Annex 17) to the coefficient of δv.

Neglecting the terms in $\delta\varrho$, δv, we have (see Annex 4)

$$\begin{aligned}
\delta P = {} & \frac{3}{2}e'\ t \\
& + \frac{3}{2}\ t\cos g', \\
& - \frac{15}{2}e' \;,,\; 2\tau \\
& + \frac{21}{4} \;,,\; 2\tau - g' \\
& - \frac{3}{4} \;,,\; 2\tau + g' \\
& + \frac{9}{2}e' \;,,\; 2g' \\
& + \frac{51}{2}e' \;,,\; 2\tau - 2g',
\end{aligned}$$

and similarly (see Annex 5),

$$\begin{aligned}
\delta Q = &\quad \frac{15}{2} e' \quad t \sin 2\tau \\
&- \frac{21}{4} \qquad ,, \ 2\tau - g' \\
&+ \frac{3}{4} \qquad ,, \ 2\tau + g' \\
&- \frac{51}{2} e' \quad ,, \ 2\tau - 2g'.
\end{aligned}$$

But in the foregoing expression of δP the terms belonging to the arguments g', $2g'$ give in $\delta \varrho$, terms which rise by integration in δv; and in forming the expressions for δP, δQ, it is proper to take account of these terms. Attending only to the terms in question, we have

$$\begin{aligned}
\frac{d^2 \delta \varrho}{dt^2} + n^2 \delta \varrho = m^2 n^2 \delta P = &\overbrace{\qquad\qquad}^{n^2 \times} \\
&\frac{3}{2} m^2 \quad t \cos g' \\
&+ \frac{9}{2} m^2 e' \quad ,, \ 2g'.
\end{aligned}$$

Now in general, if

$$\frac{d^2 \delta \varrho}{dt^2} + n^2 \delta \varrho = n^2 \quad t \cos n\alpha t,$$

then

$$\delta \varrho = \frac{2\alpha}{(1-\alpha^2)^2} \sin n\alpha t + \frac{1}{1-\alpha^2} \quad t \cos n\alpha t;$$

and hence the foregoing equation gives in $\delta \varrho$ the terms

$$\begin{aligned}
3 m^3 \quad \sin g' \qquad\qquad &+ \frac{3}{2} \qquad t \cos g' \\
+ 18 m^3 e' \ ,, \ 2g' \qquad\qquad &+ \frac{9}{2} m^2 e' \ ,, \ 2g';
\end{aligned}$$

or, neglecting the terms which contain m^3, the terms of $\delta \varrho$ are

$$\begin{aligned}
&\frac{3}{2} m^2 \quad t \cos g' \\
&+ \frac{9}{2} m^2 e' \quad ,, \ 2g'.
\end{aligned}$$

Substituting these terms in

$$\frac{d\,\delta v}{d\,t} = -\,2\,n\,\delta\varrho,$$

we have

$$\frac{d\,\delta v}{d\,t} = \overbrace{\begin{matrix} -\,3\,m^2 & t\cos g' \\ -\,9\,m^2 e' & ,,\ 2\,g'; \end{matrix}}^{n\,\times}$$

and since, in general,

$$\int t\cos n\alpha t\,dt = \frac{1}{n^2\alpha^2}\cos n\alpha t + \frac{1}{n\alpha}\,t\sin n\alpha t,$$

we obtain in δv the terms

$$\overbrace{\begin{matrix} -\,3 & \cos g' \\ -\,\frac{9}{4}e' & ,,\ 2\,g' \end{matrix}}^{n^{-1}\,\times} \quad + \quad \overbrace{\begin{matrix} -\,3\,m & t\sin g' \\ -\,\frac{9}{2}m\,e' & ,,\ 2\,g', \end{matrix}}$$

or for the present purpose the terms

$$\overbrace{\begin{matrix} -\,3 & \cos g' \\ -\,\frac{9}{4}e' & ,,\ 2\,g'; \end{matrix}}^{n^{-1}\,\times}$$

and these give in δ P the additional terms (see Annex 6)

$$\overbrace{\begin{matrix} \frac{27}{2} & \sin 2\tau \\ +\,\frac{9}{2} & ,,\ 2\tau - g' \\ +\,\frac{9}{2} & ,,\ 2\tau + g' \\ +\,\frac{153}{8}e' & ,,\ 2\tau - 2\,g' \\ +\,\frac{9}{8}\,e' & ,,\ 2\tau + 2\,g', \end{matrix}}^{n^{-1}\,\times}$$

and in δQ the additional terms (see Annex 7)

$$\begin{array}{l} \overbrace{\qquad\qquad}^{n^{-1}\times} \\ \frac{27}{2} \quad \cos 2\tau \\ + \frac{9}{2} \quad ,, \; 2\tau - g' \\ + \frac{9}{2} \quad ,, \; 2\tau + g' \\ + \frac{153}{8} e' \quad ,, \; 2\tau - 2g' \\ + \frac{9}{8} e' \quad ,, \; 2\tau + 2g'. \end{array}$$

Combining the foregoing results, we have,

$$\begin{array}{lll} \delta P = \frac{3}{2} e' \; t & & + \overbrace{\qquad\qquad}^{n^{-1}\times} \\ + \frac{3}{2} \quad t \cos g' & & 0 \quad \sin g' \\ - \frac{15}{2} e' \quad ,, \; 2\tau & & + \frac{27}{2} e' \quad ,, \; 2\tau \\ + \frac{21}{4} \quad ,, \; 2\tau - g' & & + \frac{9}{2} \quad ,, \; 2\tau - g' \\ - \frac{3}{4} \quad ,, \; 2\tau + g' & & + \frac{9}{2} \quad ,, \; 2\tau + g' \\ + \frac{9}{2} e' \quad ,, \; 2g' & & 0 \quad ,, \; 2g' \\ + \frac{51}{2} e' \quad ,, \; 2\tau - 2g' & & + \frac{153}{8} e' \quad ,, \; 2\tau - 2g' \\ 0 \quad ,, \; 2\tau + 2g' & & + \frac{9}{8} e' \quad ,, \; 2\tau + 2g', \end{array}$$

and

$$\begin{array}{lll} \delta Q = 0 \quad t \sin g' & & + \overbrace{0 \quad \cos g'}^{n^{-1}\times} \\ + \frac{15}{2} e' \quad ,, \; 2\tau & & + \frac{27}{2} e' \quad ,, \; 2\tau \\ - \frac{21}{4} \quad ,, \; 2\tau - g' & & + \frac{9}{2} \quad ,, \; 2\tau - g' \end{array}$$

$$
\begin{array}{lll}
 & & n^{-1}\times \\
+\frac{3}{4}\ t\sin(2\tau+g') & & +\frac{9}{2}\ \cos(2\tau+g') \\
0\quad ,,\ 2g' & & 0\quad ,,\ 2g' \\
-\frac{51}{2}e'\ ,,\ 2\tau-2g' & & +\frac{153}{8}e'\ ,,\ 2\tau-2g' \\
0\quad ,,\ 2\tau+2g' & & +\frac{9}{8}e'\ ,,\ 2\tau+2g';
\end{array}
$$

whence also (see Annex 8)

$$
\begin{array}{lll}
n\int\delta Q\,dt= & 0\ t\cos g' & +\overbrace{0\quad \sin g'}^{n^{-1}\times} \\
 & -\frac{15}{4}e'\ ,,\ 2\tau & +\frac{69}{8}e'\ ,,\ 2\tau \\
 & +\frac{21}{8}\ ,,\ 2\tau-g' & +\frac{15}{16}\ ,,\ 2\tau-g' \\
 & -\frac{3}{8}\ ,,\ 2\tau+g' & +\frac{39}{16}\ ,,\ 2\tau+g' \\
 & 0\ ,,\ 2g' & 0\ ,,\ 2g' \\
 & +\frac{51}{4}e'\ ,,\ 2\tau-2g' & +\frac{51}{16}e'\ ,,\ 2\tau-2g' \\
 & 0\ ,,\ 2\tau+2g' & +\frac{9}{16}e'\ ,,\ 2\tau+2g'.
\end{array}
$$

The equation for $\delta\varrho$ may be written,

$$\frac{d^2\delta\varrho}{dt^2}+n^2\delta\varrho=m^2n^2\left(\delta P+2n\int\delta Q\,dt\right),$$

and we have, (see Annex 9)

$$
\begin{array}{lll}
\delta P+2n\int\delta Q\,dt= & \frac{3}{2}e'\,t & +\overbrace{\qquad}^{n^{-1}\times} \\
 & +\frac{3}{2}\ t\cos g' & 0\quad \sin g' \\
 & -15e'\ ,,\ 2\tau & +\frac{123}{4}e'\ ,,\ 2\tau \\
 & +\frac{21}{2}\ ,,\ 2\tau-g' & +\frac{51}{8}\ ,,\ 2\tau-g'
\end{array}
$$

		$n^{-1}\times$
$-\frac{3}{2}$	$t\cos 2\tau + g'$	$+\frac{75}{8}$ $\sin 2\tau + g'$
$+\frac{9}{2}e'$	,, $2g'$	0 ,, $2g'$
$+51e'$	,, $2\tau - 2g'$	$+\frac{51}{2}e'$,, $2\tau - 2g'$
0	,, $2\tau + 2g'$	$+\frac{9}{4}e'$,, $2\tau + 2g'$.

Hence observing that a term $n^2 t\cos n\alpha t$ in $\frac{d^2\delta\varrho}{\delta t^2} + n^2\delta\varrho$, gives in $\delta\varrho$ the terms

$$\frac{1}{1-\alpha^2}\, t\cos n\alpha t \qquad + \frac{2\alpha}{(1-\alpha^2)^2}\,\frac{1}{n}\sin n\alpha t,$$

and a term $n\sin n\alpha t$ in $\frac{d^2\delta\varrho}{dt^2} + n^2\delta\varrho$, gives in $\delta\varrho$ the term

$$\frac{1}{1-\alpha^2}\,\frac{1}{n}\sin n\alpha t,$$

we have (see Annex 10, but restoring the factor f'),

$\delta\varrho = f'$ into as follows, viz.

		$n^{-1}\times$
$\frac{3}{2}m^2e'\,t$		+
$+\frac{3}{2}m^2$	$t\cos g'$	$3m^3$ $\sin g'$
$+5m^2e'$	,, 2τ	$-\frac{203}{12}m^2e'$,, 2τ
$-\frac{7}{2}m^2$	,, $2\tau - g'$	$+\frac{61}{24}m^2$,, $2\tau - g'$
$+\frac{1}{2}m^2$	,, $2\tau + g'$	$-\frac{91}{24}m^2$,, $2\tau + g'$
$+\frac{9}{2}m^2e'$	,, $2g'$	$-18m^3e'$,, $2g'$
$-17m^2e'$	,, $2\tau - 2g'$	$+\frac{85}{6}m^2e'$,, $2\tau - 2g'$
0	,, $2\tau + 2g'$	$-\frac{3}{4}m^2e'$,, $2\tau + 2g'$.

The first column, containing the term in t and the terms t cos arg., shows that the constant term of ϱ, and the terms involving the cosines of the same arguments, as obtained without attending to the variation of the solar excentricity, are correct as regards the first power of t, when for the constant excentricity e' we substitute $e' + f' t$. In fact, the above-mentioned expression (correct to e'^2) of $\frac{1}{\varrho}$ gives

$$\begin{aligned}
\varrho = 1 &- \tfrac{1}{6} m^2 + \tfrac{3}{4} m^2 e'^2 \\
&+ \tfrac{3}{2} m^2 e'^2 && \cos g' \\
&- m^2 + \tfrac{5}{2} m^2 e'^2 && ,, \; 2\tau \\
&- \tfrac{7}{2} m^2 e' && ,, \; 2\tau - g' \\
&+ \tfrac{1}{2} m^2 e' && ,, \; 2\tau + g' \\
&+ \tfrac{9}{4} m^2 e'^2 && ,, \; 2g' \\
&- \tfrac{17}{2} m^2 e'^2 && ,, \; 2\tau - 2g' \\
&0 && ,, \; 2\tau + 2g',
\end{aligned}$$

and putting therein $e' + f' t$ in the place of e', we have the first column of the foregoing expression of $\delta \varrho$.

The second column, involving sin arg., contains the new periodic terms considered in Prof. Adams' Memoir of 1853, and the coefficients for the arguments g', 2τ, $2\tau - g'$, $2\tau + g'$, agree with his values; observing that his terms belong to $\delta \frac{1}{\varrho} = -\frac{\delta \varrho}{\varrho^2} = -\delta \varrho$, so that the signs are reversed; those for the remaining arguments $2g'$, $2\tau - 2g'$, $2\tau + 2g'$, are not given by him.

The equation for δv may be written,

$$\frac{d\,\delta v}{dt} = -2n\,\delta\varrho + m^2 n^2 \int \delta Q\, dt,$$

and we have (see Annex 11)

$$\begin{aligned}
\frac{d\,\delta v}{dt} = &\overbrace{\qquad\qquad\qquad}^{n \times} \\
&- 3 m^2 e' \quad t && \\
&- 3 m^2 \quad t \cos g' && - 6 m^3 \quad \sin g \\
&+ \tfrac{55}{4} m^2 e' \quad ,, \; 2\tau && + \tfrac{1019}{24} m^2 e' \quad ,, \; 2\tau
\end{aligned}$$

$$n \times$$

$$+\frac{77}{8}m^2 \quad t\cos 2\tau - g' \qquad -\frac{199}{48}m^2 \quad \sin 2\tau - g'$$

$$-\frac{11}{8}m^2 \quad ,, \; 2\tau + g' \qquad +\frac{481}{48}m^2 \quad ,, \; 2\tau + g'$$

$$-9\,m^2 e' \quad ,, \; 2g' \qquad -36\,m^3 e' \quad ,, \; 2g'$$

$$+\frac{187}{8}m^2 e' \quad ,, \; 2\tau - 2g' \qquad -\frac{1207}{48}m^2 e' \quad ,, \; 2\tau - 2g'$$

$$0 \quad ,, \; 2\tau + 2g' \qquad +\frac{33}{16}m^2 e' \quad ,, \; 2\tau + 2g',$$

whence, integrating by the formulæ

$$\int t\cos n\alpha t \; dt = \frac{1}{n\alpha}\, t\sin n\alpha t + \frac{1}{n^2\alpha^2}\cos n\alpha t,$$

$$\int \sin n\alpha t \; dt = -\frac{1}{n\alpha}\cos n\alpha t,$$

we have (see Annex 12, but restoring the factor f'),

$\delta v = f'$ into as follows, viz.

$$-\frac{3}{2}\,m^2 n\, e'\, t^2$$

$$n^{-1} \times$$

$$-3m \quad t\sin g' \qquad +\left(-3+6m^2\right)\cos g'$$

$$-\frac{55}{8}m^2 e' \quad ,, \; 2\tau \qquad -\frac{74}{3}m^2 e' \quad ,, \; 2\tau$$

$$+\frac{77}{16}m^2 \quad ,, \; 2\tau - g' \qquad +\frac{215}{48}m^2 \quad ,, \; 2\tau - g'$$

$$-\frac{11}{16}m^2 \quad ,, \; 2\tau + g' \qquad -\frac{257}{48}m^2 \quad ,, \; 2\tau + g'$$

$$-\frac{9}{2}m e' \quad ,, \; 2g' \qquad \left(-\frac{9}{4}+18m^2\right)e' \quad ,, \; 2g'$$

$$+\frac{187}{8}m^2 e' \quad ,, \; 2\tau - 2g' \qquad +\frac{2239}{96}m^2 e' \quad ,, \; 2\tau - 2g'$$

$$0 \quad ,, \; 2\tau + 2g' \qquad -\frac{33}{32}m^2 e' \quad ,, \; 2\tau + 2g'.$$

The first column, containing t sin arg., may be obtained from the before-mentioned expression (accurate to e'^2) of v, by substituting therein $e' + f' t$ in the place of e'.

The term $-\frac{3}{2}m^2\,n\,e'f'\,t^2$; or, as it may be written, $-\frac{3}{2}m^2\int n\left[\left(e'+f't\right)^2-e'^2\right]dt$, is the first term of the acceleration; the other terms of the second column are the new periodic terms in δv, considered by Prof. Adams; the coefficients for the arguments g', 2τ, $2\tau-g'$, $2\tau+g'$, agreeing with his values, but those for the remaining arguments $2g'$, $2\tau-2g'$, $2\tau+2g'$ not being given by him.

III.

Proceeding now to the calculation of the term in m^4 of the acceleration, we have,

$$\frac{d\,\delta v}{dt}=-\frac{2}{\varrho}\frac{dv}{dt}\delta\varrho+\frac{m^2n^2}{\varrho^2}\left(\mathrm{C}+\int\delta\mathrm{Q}\,dt\right),$$

where the non-periodic part of $\delta\varrho$ is of the form,

$$\delta\varrho=\left(\frac{3}{2}m^2+\square\,m^4\right)e'f'\,t,$$

and it is in the first place necessary to find the value of the numerical coefficient $\square$, in fact to calculate the secular part of $\delta\varrho$ as far as m^4. Reverting to the equation

$$\frac{d^2\,\delta\varrho}{dt^2}+n^2\delta\varrho=$$

$$\left(n^2+\frac{2n^2}{\varrho^3}-3\left(\frac{dv}{dt}\right)^2\right)\delta\varrho+m^2n^2\left(\delta\mathrm{P}+\frac{2}{\varrho}\frac{dv}{dt}\left(\mathrm{C}+\int\delta\mathrm{Q}\,dt\right)\right),$$

and as before omitting in the process the factor f'.

The part $\left(n^2+\frac{2n^2}{\varrho^3}-3\left(\frac{dv}{dt}\right)^2\right)\delta\varrho$ contains (see Annex 13), a term

$$=\frac{381}{8}m^4n^2e'\,t.$$

The part of $m^2n^2\,\delta\mathrm{P}$, which involves $\delta\varrho$, contains (see Annex 14) a term

$$=-\frac{495}{32}m^4n^2e'\,t.$$

The part of $m^2 n^2 \delta P$, which depends on δv, contains a term

$$= -\frac{15}{4} m^4 n^2 e' t.$$

The part of $m^2 n^2 \delta P$, depending on $\delta \varrho'$ and $\delta v'$, is found (see Annex 18) to contain, besides the term $\frac{3}{2} m^2 n e' t$ in m^2 already obtained, a new term

$$= -\frac{647}{32} m^4 n^2 e' t.$$

And finally the part $m^2 n^2 \frac{2}{\varrho} \frac{dv}{dt} \left(C + \int \delta Q \, dt \right)$ is found (see Annex 19) to contain a term

$$\left(2 \cdot \frac{-1455}{32} + \frac{675}{32} = \right) - \frac{2235}{32} m^4 n^2 e' t.$$

where the component coefficient $-\frac{1455}{32}$, which arises from the new periodic terms of $\delta \varrho$ and δv is separately calculated (see Annex 21).

Hence $\frac{d^2 \delta \varrho}{dt^2} + n^2 \delta \varrho$ contains the term

$$\left(\square = \frac{381}{8} - \frac{495}{32} - \frac{15}{4} - \frac{647}{32} - \frac{2235}{32} = \right) - \frac{1973}{32} m^4 n^2 e' t,$$

and this gives in $\delta \varrho$ the term

$$-\frac{1973}{32} m^4 e' t,$$

so that, restoring the term in m^2, and the common factor f', the complete secular term of $\delta \varrho$ is

$$\delta \varrho = \left(\frac{3}{2} m^2 - \frac{1973}{32} m^4 \right) e' f' t,$$

which, as will be shown, Art. IV., agrees with Prof. Adams' result.

Resuming now the equation,

$$\frac{d \delta v}{dt} = -\frac{2}{\varrho} \frac{dv}{dt} \delta \varrho + \frac{m^2 n^2}{\varrho^2} \left(C + \int \delta Q \, dt \right),$$

the part $-\frac{2}{\varrho}\frac{dv}{dt}\delta\varrho$ contains (see Annex 22) the term

$$\left(\frac{1973}{16}+\frac{275}{8}=\right)\frac{2523}{16}m^4 n e' t,$$

and the part $\frac{m^2n^2}{\varrho^2}\left(C+\int\delta Q\,dt\right)$ contains (see Annex 23) the term

$$\left(\frac{45}{8}+\frac{1455}{32}=\right)-\frac{1275}{32}m^4 n e' t,$$

so that we have in $\frac{d\delta v}{dt}$ the term

$$\left(\frac{2523}{16}-\frac{1275}{32}=\right)\frac{3771}{32}m^4 n e' t,$$

giving in δv the term

$$\frac{3771}{64}m^4 n e' t^2,$$

or, restoring the term in m^2 and the common factor f', the complete secular term of δv is

$$\delta v=\left(-\frac{3}{2}m^2+\frac{3771}{64}m^4\right)n e' f' t^2,$$

which agrees with the value obtained by Prof. Adams. It is right to remark that the m of Prof. Adams is different from that of the present Memoir; we have in fact,

$$m\ (\text{Adams})=m\left\{1+\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)2e'f't\right\}$$

in the notation of the present Memoir; but as f'^2 is throughout neglected, we may in the foregoing expression of the secular part of δv substitute the m of Prof. Adams. And then if the term be written in the form

$$\delta v=\left(-\frac{3}{2}m^2+\frac{3771}{64}m^4\right)\int\left[\left(e'+f't\right)^2-e'^2\right]n\,dt,$$

the two results are seen to agree together. But as regards the before-mentioned secular term,

$$\delta\varrho = \left(\frac{3}{2}m^2 - \frac{1973}{32}m^4\right)e'f't,$$

the identification is less easy, and I shall consider it in the following article.

IV.

It will be convenient to write M, N, A, E′, in place of the m, n, a, e', of the foregoing part of the present Memoir, and to now use m, n, e', in the significations in which they are employed by Prof. Adams; E′ (the constant part of the solar excentricity) is his E′, and his e' is $E' + f't$. As to his symbols a, $a_{,}$, these, I think, ought to have been represented, and I shall here represent them by a, a$_{,}$.* And I take a such that $n^2 a^3 =$ Sum of the masses of the Earth and Moon; or, taking this to be unity, we have $N^2 A^3 = 1$, $n^2 a^3 = 1$.

The formulæ of Prof. Adams' memoir, which it will be necessary to make use of, may be written

$$\frac{a}{r} = 1 - \frac{11}{8}m^2 - \frac{201}{16}m^4e'^2$$

$$- \frac{3}{2}m^2e' \qquad \cos g'$$

$$+ m^2 - \frac{5}{2}m^2e'^2 \quad ,, \quad 2\tau$$

$$+ \frac{7}{2}m^2e' \qquad ,, \quad 2\tau - g'$$

$$- \frac{1}{2}m^2e' \qquad ,, \quad 2r + g'$$

the sine terms being disregarded.

$$n = N\left\{1 + \left(-\frac{3}{2}m^2 + \frac{3771}{64}m^4\right)2E'f't\right\},$$

(whence also

$$m = M\left\{1 + \left(-\frac{3}{2}m^2 - \frac{3771}{64}m^4\right)2E'f't\right\}$$

* Plana, in his Lunar Theory, uses the three letters a, a$_{,}$, a; his a and n being such that $n^2 a^3 =$ Sum of the masses of the Earth and Moon. There is an obvious inconvenience in writing a, $a_{,}$, in the place of his a, a$_{,}$.

since $m = \frac{n'}{n}$, $\mathrm{M} = \frac{\mathrm{N}'}{\mathrm{N}}$).

$$\frac{1}{n} = a_{,}^{\frac{3}{2}} \left\{ 1 + m^2 - \frac{197}{64} m^4 + \left(\frac{3}{2} m^2 - \frac{3867}{64} m^4 \right) e'^2 \right\},$$

$$1 = \frac{a}{a_{,}} \left\{ 1 - \frac{1}{2} m^2 + \frac{13}{4} m^4 + \left(-\frac{3}{4} m^2 + \frac{3201}{64} m^4 \right) e'^2 \right\},$$

which formulæ, observing that $\frac{1}{\varrho + \delta\varrho} = \frac{A}{r}$ lead to the value of $\delta\varrho$, and we should obtain for the secular part the foregoing expression, which will now be

$$\delta\varrho = \left(\frac{3}{2} \mathrm{M}^2 - \frac{1973}{32} \mathrm{M}^4 \right) \mathrm{E}' f' t.$$

We in fact have

$$a^{\frac{3}{2}} = \frac{1}{n} = a_{,}^{\frac{3}{2}} \left\{ 1 + m^2 - \frac{197}{64} m^4 + \left(\frac{3}{2} m^2 - \frac{3867}{64} m^4 \right) e'^2 \right\},$$

And thence

$$\begin{aligned} a &= a_{,} \left\{ 1 + \frac{2}{3} m^2 - \frac{197}{96} m^4 + \left(m^2 - \frac{3867}{96} m^4 \right) e'^2 \right. \\ &\qquad \left. - \frac{1}{9} \left(\quad m^4 \qquad\qquad + 3\, m^4\, e'^2 \right) \right\} \\ &= a_{,} \left\{ 1 + \frac{2}{3} m^2 - \frac{623}{288} m^4 + \left(m^2 - \frac{3899}{96} m^4 \right) e'^2 \right\}, \end{aligned}$$

but

$$1 = \frac{a}{a_{,}} \left\{ 1 - \frac{1}{2} m^2 + \frac{13}{4} m^4 + \left(-\frac{3}{4} m^2 + \frac{3201}{64} m^4 \right) e'^2 \right\};$$

and therefore

$$\begin{aligned} a = a \left\{ 1 \right. &+ \frac{2}{3} m^2 - \frac{623}{288} m^4 + \left(m^2 - \frac{3899}{96} m^4 \right) e'^2 \\ &- \frac{1}{2} m^2 + \frac{13}{4} m^4 \qquad - \frac{1}{2} m^4\, e'^2 \\ &- \frac{1}{3} m^4 + \left(-\frac{3}{4} m^2 + \frac{3201}{64} m^4 \right) e'^2 \\ &- \frac{1}{2} m^4\, e'^2 \Big\} \\ = a \left\{ 1 \right. &+ \frac{1}{6} m^2 + \frac{217}{288} m^4 + \left(\frac{1}{4} m^2 + \frac{1613}{192} m^4 \right) e'^2 \Big\}; \end{aligned}$$

and hence, since for the non-periodic part

$$\frac{a}{r} = 1 - \frac{11}{8} m^4 - \frac{201}{16} m^4 e'^2,$$

we find

$$\frac{a}{r} = 1 + \frac{1}{6} m^2 + \frac{217}{288} m^4 + \left(\frac{1}{4} m^2 + \frac{1613}{192} m^4\right) e'^2$$

$$- \frac{11}{8} m^4 \qquad - \frac{201}{16} m^4 e'^2$$

$$= 1 + \frac{1}{6} m^2 - \frac{179}{288} m^4 + \left(\frac{1}{4} m^2 - \frac{799}{192} m^4\right) e'^2$$

$$= 1 + \frac{1}{6} m^2 - \frac{179}{288} m^4 + \left(\frac{1}{4} m^2 - \frac{799}{192} m^4\right) \mathrm{E}'^2 + \left(\frac{1}{4} m^2 - \frac{799}{192} m^4\right) 2\, \mathrm{E}' f' t.$$

But

$$\frac{\mathrm{A}}{a} = \left(\frac{n}{\mathrm{N}}\right)^{\frac{2}{3}} = 1 + \left(-m^2 + \frac{3771}{96} m^4\right) 2\, \mathrm{E}' f' t;$$

and therefore

$$\frac{\mathrm{A}}{r} = 1 + \frac{1}{6} m^2 - \frac{179}{288} m^4 + \left(\frac{1}{4} m^2 - \frac{799}{192} m^4\right) \mathrm{E}'^2 + \left(\frac{1}{4} m^2 - \frac{799}{192} m^4\right) 2\, \mathrm{E}' f' t$$

$$+ \left(-m^2 + \frac{3771}{96} m^4\right) 2\, \mathrm{E}' f' t$$

$$- \frac{1}{6} m^4 . 2\, \mathrm{E}' f' t$$

$$= 1 + \frac{1}{6} m^4 - \frac{179}{288} m^4 + \left(\frac{1}{4} m^2 - \frac{799}{192} m^4\right) \mathrm{E}'^2 + \left(-\frac{3}{4} m^2 + \frac{2237}{64} m^4\right) 2\, \mathrm{E}' f' t.$$

But we have

$$m = \mathrm{M}\left\{1 + \left(\frac{3}{2} \mathrm{M}^2 - \frac{3771}{64} \mathrm{M}^4\right) 2\, \mathrm{E}' f' t\right\};$$

and thence in the foregoing expression

$$m^2 = \mathrm{M}^2 + 3\, \mathrm{M}^4 . 2\, \mathrm{E}' f' t.$$

$$m^4 = \mathrm{M}^4;$$

and therefore

$$\frac{A}{r} = 1 + \frac{1}{6}M^2 - \frac{179}{288}M^4 + \left(\frac{1}{4}M^2 - \frac{799}{192}M^4\right)E'^2 + \left(-\frac{3}{4}M^2 + \frac{2237}{64}M^4\right) 2E'f't + \frac{1}{2}M^4 \cdot 2E'f't$$

$$= 1 + \frac{1}{6}M^2 - \frac{179}{288}M^4 + \left(\frac{1}{4}M^2 - \frac{799}{192}M^4\right)E'^2 + \left(-\frac{3}{4}M^2 + \frac{2269}{64}M^4\right) 2E'f't;$$

and observing that in the periodic terms we may write A, in the place of a, and neglect the sine terms, we have

$$\frac{A}{r} = \frac{1}{\varrho + \delta\varrho} = 1 + \frac{1}{6}M^2 - \frac{179}{288}M^4 + \left(\frac{1}{4}M^2 - \frac{799}{192}M^4\right)E'^2 + \left(-\frac{3}{4}M^2 + \frac{2269}{64}M^4\right) 2E'f't$$

$$-\frac{3}{2}m^2 e' \quad \cos g'$$

$$+ m^2\left(1 - \frac{5}{2}e'^2\right) \quad ,, \quad 2\tau$$

$$+\frac{7}{2}m^2 e' \quad ,, \quad 2\tau - g'$$

$$-\frac{1}{2}m^2 e' \quad ,, \quad 2\tau + g',$$

say $\frac{1}{\varrho + \delta\varrho} = 1 + X$; and thence $\varrho + \delta\varrho = \frac{1}{1+X} = 1 - X + X^2$, the non-periodic part whereof is

$$1 - \frac{1}{6}M^2 + \frac{179}{128}M^4 + \left(-\frac{1}{4}M^2 + \frac{179}{192}M^4\right)E'^2 + \left(\frac{3}{4}M^2 - \frac{2269}{64}M^4\right) 2E'f't$$

$$+\frac{1}{36}M^4 \qquad +\frac{1}{12}M^4 E'^2 \qquad +\frac{1}{4}M^4 \cdot 2E'f't$$

$$+\frac{1}{2}\frac{9}{4}m^4 e'^2$$

$$+\frac{1}{2} \quad m^4\left(1 - 5e'^2\right)$$

$$+\frac{1}{2}\frac{49}{4}m^4 e'^2$$

$$+\frac{1}{2}\frac{1}{4}m^4 e'^2,$$

where the terms in m^4 and $m^4 e'^2$ are

$$= \tfrac{1}{2} m^4 + \left(\tfrac{9}{8} - \tfrac{5}{2} + \tfrac{49}{8} + \tfrac{1}{8} =\right) \tfrac{39}{8} m^4 e'^2,$$

which are

$$= \tfrac{1}{2} \mathrm{M}^4 \qquad + \tfrac{39}{8} \mathrm{M}^4 \mathrm{E}'^2 \qquad + \tfrac{39}{8} \mathrm{M}.^4 \; 2 \mathrm{E}' f' t.$$

so that the foregoing expression of the non-periodic part of $\varrho + \delta\varrho$ is

$$= 1 - \tfrac{1}{6} \mathrm{M}^2 + \left(\tfrac{179}{128} + \tfrac{1}{36} + \tfrac{1}{2} =\right) \tfrac{2219}{1152} \mathrm{M}^4$$

$$+ \left(-\tfrac{1}{4} \mathrm{M}^2 + \left(\tfrac{179}{192} + \tfrac{1}{12} + \tfrac{39}{8} =\right) \tfrac{1131}{192} \mathrm{M}^4\right) \mathrm{E}'^2$$

$$+ \left(\tfrac{3}{4} \mathrm{M}^2 + \left(-\tfrac{2269}{64} - \tfrac{1}{4} + \tfrac{39}{8} =\right) - \tfrac{1973}{64} \mathrm{M}^4\right) 2 \mathrm{E}' f' t;$$

or the secular term of $\delta\varrho$ is

$$= \left(\tfrac{3}{4} \mathrm{M}^2 - \tfrac{1973}{64} \mathrm{M}^4\right) 2 \mathrm{E}' f' t,$$

which is the required formula.

V.

It is interesting to see how the coefficient $\frac{3771}{64}$ is made up. In Prof. Adams' Memoir we have

$$\tfrac{3771}{64} =$$

$$= -\tfrac{3}{2} - \tfrac{3}{4} \qquad \left(= -\tfrac{9}{4}\right)$$

$$+ \tfrac{15}{4} + \tfrac{135}{64} - \tfrac{117}{8} + \tfrac{495}{128} - \tfrac{285}{16} - \tfrac{147}{16} - \tfrac{3}{16} - \tfrac{1323}{256} - \tfrac{27}{256}$$

$$\left(= -\tfrac{2391}{64}\right)$$

$$+ \tfrac{9}{2} - \tfrac{27}{4} - \tfrac{45}{2} - \tfrac{45}{2} - \tfrac{15}{2} - \tfrac{495}{32} + \tfrac{285}{4} - \tfrac{147}{4} - \tfrac{3}{4} + \tfrac{441}{8} + \tfrac{441}{8} + \tfrac{9}{8} + \tfrac{9}{8}$$

$$\left(= +\tfrac{3153}{32}\right),$$

where it may be remarked that the terms

$$\frac{495}{128} - \frac{285}{16} = \left(-\frac{1785}{128}\right)$$

and

$$-\frac{495}{32} + \frac{285}{4} = \left(+4 \cdot \frac{1785}{128}\right)$$

make together $3 \cdot \frac{1785}{128} = \frac{5355}{128}$, and that it is in fact by the addition of these terms that Plana's coefficient $\frac{2187}{128}$ is changed into $\frac{3771}{64}$.

But in the present Memoir the coefficient $\frac{3771}{64}$ is obtained by means of an entirely different set of component numbers, viz. we have

$$\frac{3771}{64} = -\frac{381}{8} + \frac{495}{32} + \frac{15}{4} + \frac{647}{32} + \frac{1455}{16} - \frac{675}{32} + \frac{275}{16} + \frac{45}{16} - \frac{1455}{64}.$$

I had imagined, from the way in which the numbers $\frac{1455}{16} - \frac{1455}{64}$ presented themselves, that, if they were omitted, Plana's value $\frac{2187}{128}$ would have been obtained; but the result shows that this is not so.

As just deduced from the formula of Prof. Adams, the number $\frac{1973}{64}$ is obtained as follows, viz.

$$\frac{1973}{64} =$$

$$\left(-\frac{3771}{96} - 1\right) - \frac{1}{3} - \frac{1}{2} + \left(\frac{3153}{64} + \frac{3}{4}\right) - \frac{1}{2} - \frac{201}{16} + \frac{3771}{96} - \frac{1}{6} - \frac{1}{2} + \frac{1}{4} - \frac{39}{8}$$

$$= -1 - \frac{1}{3} - \frac{1}{2} + \frac{3}{4} - \frac{1}{2} - \frac{1}{6} - \frac{1}{2} + \frac{1}{4} \qquad \left(= -1\right)$$

$$+ \frac{3153}{64} - \frac{201}{16} - \frac{39}{8},$$

where *ut suprà*

$$\frac{3153}{64} =$$

$$\frac{1}{2}\left\{\frac{9}{2} - \frac{27}{4} - \frac{45}{2} - \frac{45}{2} - \frac{15}{2} - \frac{495}{32} + \frac{285}{4} - \frac{147}{4} - \frac{3}{4} + \frac{441}{8} + \frac{441}{8} + \frac{9}{8} + \frac{9}{8}\right\},$$

$-\frac{201}{16}$ is a number occurring in his Memoir, and which is in effect obtained irrespectively of the new periodic terms, and $-\frac{39}{8}$ is a number obtained as above, irrespectively of the new periodic terms. According to the method of the present Memoir, the number $\frac{1973}{64}$ was obtained in the form

$$\frac{1973}{64} = -\frac{1}{2}\left(\frac{381}{8} - \frac{495}{32} - \frac{15}{4} - \frac{647}{32} - \frac{1455}{16} + \frac{675}{32}\right).$$

VI.

If the investigation were pursued further, a question would arise as to the proper form to be given to the arguments; for in these, $nt+\varepsilon$ seems to stand in the place of v, the value whereof is

$$v = nt + \varepsilon - \left(\frac{3}{2}m^2 - \frac{3771}{64}m^4\right) ne'f't^2,$$

say $v = nt + \varepsilon + kne'f't^2$, and it might be considered that in the arguments $nt+\varepsilon$ should be changed into $nt+\varepsilon+kne'f't^2$, or, what is the same thing, that τ should be changed into $\tau+kne'f't^2$, but that g' should remain unaltered (this assumes that there is not in the Sun's longitude any term corresponding to the acceleration). The arguments, instead of being of the simple form kt, would thus be of the form $kt+k_2f't^2$. But this would not only increase the difficulty of integration, but would be inconsistent with the general plan of the solution; and it would seem to be the proper course to imagine the cosine or sine of such an argument to be developed $\left(\begin{matrix}\cos\\ \sin\end{matrix} kt + k_2f't^2 = \begin{matrix}\cos\\ \sin\end{matrix} kt \mp k_2f't^2 \begin{matrix}\sin\\ \cos\end{matrix} kt\right)$ in such manner as to bring the secular part of the argument outside the cos or sin; this is, in fact, the form which the solution takes when the arguments are left throughout in their original form, for the terms of the form $f't^2 \begin{matrix}\cos\\ \sin\end{matrix}$ arg. would present themselves in the subsequent approximations. But I shall not at present further examine the question.

Annexes containing the Details of the Calculation.

Annex 1.

Calculation of part of δP.

$$\delta P = \frac{1}{\varrho'^3}\left(\frac{1}{2} + \frac{3}{2}\cos 2v - 2v'\right)\delta\varrho$$

$$+ \frac{\varrho}{\varrho'^3}\left(-3\sin 2v - 2v'\right)\delta v.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$\frac{1}{2} + \frac{3}{2}\cos 2v - 2v' =$	$\frac{1}{\varrho'^3} =$
$\frac{1}{2}$	1
$+\frac{3}{2}\cos 2\tau$	$+ 3e' \cos g'.$
$+ 3e' \,,, 2\tau - g'$	
$- 3e' \,,, 2\tau + g'$	

Product is =

$$\frac{1}{2} \qquad + \frac{3}{2}e' \cos g'$$

$$+\frac{3}{2}\cos 2\tau \quad + \frac{9}{2}e'\left(\frac{1}{2}\cos 2\tau - g' + \frac{1}{2}\cos 2\tau + g'\right)$$

$$+ 3e' \,,, 2\tau - g'$$

$$- 3e' \,,, 2\tau + g'$$

$$= \frac{1}{2}$$

$$+ \frac{3}{2}e' \cos g'$$

$$+ \frac{3}{2} \,,, 2\tau$$

$$\left(+3+\frac{9}{4} =\right) + \frac{21}{4}e' \,,, 2\tau - g'$$

$$\left(-3+\frac{9}{4} =\right) - \frac{3}{4}e' \,,, 2\tau + g'.$$

which is the coefficient of $\delta\varrho$.

And

$-3\sin 2v - 2v' =$	$\frac{1}{\varrho'^3} =$
$-3 \quad \sin 2\tau$	1
$-6e' \;,,\; 2\tau - g'$	$+3e'\cos g'.$
$+6e' \;,,\; 2\tau + g'$	

Product is $=$

$$\begin{aligned} &-3 \quad \sin 2\tau && -9e'\left(\tfrac{1}{2}\sin 2\tau - g' + \tfrac{1}{2}\sin 2\tau + g'\right) \\ &-6e' \;,,\; 2\tau - g' \\ &+6e' \;,,\; 2\tau + g' \\ &= && -3 \quad \sin 2\tau \\ &\left(-6 - \tfrac{9}{2} =\right) && -\tfrac{21}{2}e' \;,,\; 2\tau - g' \\ &\left(+6 - \tfrac{9}{2} =\right) && +\tfrac{3}{2}e' \;,,\; 2\tau + g'; \end{aligned}$$

or, since $\varrho = 1$, this is the coefficient of δv.

Annex 2.

Calculation of part of δQ.

$$\delta Q = \frac{\varrho}{\varrho'^3}(-3\sin 2v - 2v')\,\delta\varrho$$

$$+\frac{\varrho^2}{\varrho'^3}(-3\cos 2v - 2v')\,\delta v.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$-3\sin 2v - 2v' =$	$\frac{1}{\varrho'^3} =$
$-3 \quad \sin 2\tau$	1
$-6e' \;,,\; 2\tau - g'$	$+3e'\cos g'.$
$+6e' \;,,\; 2\tau + g'$	

Product is =

$$-3 \sin 2\tau \qquad -9e'\left(\tfrac{1}{2}\sin 2\tau - g' + \tfrac{1}{2}\sin 2\tau + g'\right)$$

$$-6e' \;,,\; 2\tau - g'$$

$$-6e' \;,,\; 2\tau + g'$$

$$= \qquad -3 \sin 2\tau$$

$$\left(-6 - \tfrac{9}{2} =\right) \quad -\tfrac{21}{2}e' \;,,\; 2\tau - g'$$

$$\left(+6 - \tfrac{9}{2} =\right) \quad +\tfrac{3}{2}e' \;,,\; 2\tau + g';$$

or, since $\varrho = 1$, this is the coefficient of $\delta\varrho$.

$-3\cos 2v - 2v' =$	$\frac{1}{\varrho'^3} =$
$-3 \cos 2\tau$	1
$-6e' \;,,\; 2\tau - g'$	$+3e' \cos g'$
$+6e' \;,,\; 2\tau + g'$	

Product is =

$$-3 \cos 2\tau \qquad -9e'\left(\tfrac{1}{2}\cos 2\tau - g' + \tfrac{1}{2}\cos 2\tau + g'\right)$$

$$-6e' \;,,\; 2\tau - g'$$

$$+6e' \;,,\; 2\tau + g'$$

$$= \qquad -3 \cos 2\tau$$

$$\left(-6 - \tfrac{9}{2} =\right) \quad -\tfrac{21}{2}e' \;,,\; 2\tau - g'$$

$$\left(+6 - \tfrac{9}{2} =\right) \quad +\tfrac{3}{2}e' \;,,\; 2\tau + g';$$

or, since $\varrho^2 = 1$, this is the coefficient of δv.

Annex 3.

Calculation of $\begin{matrix}\cos\\ \sin\end{matrix} 2v - 2v'$.

$$v - v' = \tau - 2e' \sin g'$$

$$2v - 2v' = 2\tau - 4e' \sin g'$$

$$\cos 2v - 2v' = \cos 2\tau$$
$$+ \sin 2\tau \cdot 4e' \sin g'$$
$$= \cos 2\tau$$
$$+ 2e' \;,, \; 2\tau - g'$$
$$- 2e' \;,, \; 2\tau + g'$$

$$\sin 2v - 2v' = \sin 2\tau$$
$$- \cos 2\tau \cdot 4e' \sin g'$$
$$= \sin 2\tau$$
$$+ 2e' \;,, \; 2\tau - g'$$
$$- 2e' \;,, \; 2\tau + g'$$

The expressions are calculated (*post*, Annex 16) as far as m^2.

Annex 4.

Calculation of a part of δP.

$$\delta P = \varrho \left\{ \left(\frac{1}{2} + \frac{3}{2} \cos 2v - 2v' \right) \delta \frac{1}{\varrho'^3} + \left(3 \sin 2v - 2v' \right) \frac{\delta v'}{\varrho'^3} \right\}.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$\frac{1}{2} + \frac{3}{2} \cos 2v - 2v' =$	$\delta \frac{1}{\varrho'^3} =$
$\frac{1}{2}$	$3e' t$
$+ \frac{3}{2} \cos 2\tau$	$+ 3 \quad t \cos g'$
$+ 3e' \;,, \; 2\tau - g'$	$+ 9e' \quad ,, \; 2g'.$
$- 3e' \;,, \; 2\tau + g'$	

Product is =

$$\begin{aligned}
&\tfrac{3}{2}\, e' \quad t \\
+\ &\tfrac{9}{2}\, e' \quad t \cos 2\tau \\
+\ &\tfrac{3}{2} \quad ,, \ g' \\
+\ &\tfrac{9}{2} \left(\tfrac{1}{2} t \cos 2\tau - g' \quad + \tfrac{1}{2} t \cos 2\tau + g' \right) \\
+\ &9\, e' \left(\tfrac{1}{2} \quad ,, \ 2\tau - 2g' + \tfrac{1}{2} \quad ,, \ 2\tau \right) \\
-\ &9\, e' \left(\tfrac{1}{2} \quad ,, \ 2\tau \quad + \tfrac{1}{2} \quad ,, \ 2\tau + 2g' \right) \\
+\ &\tfrac{9}{2} e' \ t \cos 2g' \\
+\ &\tfrac{27}{2} e' \left(\tfrac{1}{2} t \cos 2\tau - 2g' \quad + \tfrac{1}{2} t \cos 2\tau + 2g' \right).
\end{aligned}$$

which is =

$$\begin{aligned}
&\tfrac{3}{2} e' \ t \\
+\ &\tfrac{3}{2} \quad t \cos g' \\
\left(\tfrac{9}{2} + \tfrac{9}{2} - \tfrac{9}{2} = \right) \ + \ &\tfrac{9}{2} e' \quad ,, \ 2\tau \\
+\ &\tfrac{9}{4} \quad ,, \ 2\tau - g' \\
+\ &\tfrac{9}{4} \quad ,, \ 2\tau + g' \\
+\ &\tfrac{9}{2} e' \quad ,, \ 2g' \\
\left(\tfrac{9}{2} + \tfrac{27}{4} = \right) \ + \ &\tfrac{45}{4} e' \quad ,, \ 2\tau - 2g' \\
\left(-\tfrac{9}{2} + \tfrac{27}{4} = \right) \ + \ &\tfrac{9}{4} e' \quad ,, \ 2\tau + 2g'
\end{aligned}$$

$3 \sin 2v - 2v' =$

$$\begin{aligned}
&3 \quad \sin 2\tau \\
+\ &6\, e' \ ,, \ 2\tau - g' \\
-\ &6\, e' \ ,, \ 2\tau + g'
\end{aligned}$$

$$\frac{\delta v'}{e'^3} =$$

$$\begin{aligned}
&2 \quad t \sin g' \\
+\ &\tfrac{11}{2} e' \ ,, \ 2g'
\end{aligned}$$

Product is =

$$6\left(\tfrac{1}{2}\,t\cos 2\tau - g' - \tfrac{1}{2}\,t\cos 2\tau + g'\right)$$

$$+12e'\left(\tfrac{1}{2}\ ,,\ 2\tau - 2g' - \tfrac{1}{2}\ ,,\ 2\tau\right)$$

$$-12e'\left(\tfrac{1}{2}\ ,,\ 2\tau - \tfrac{1}{2}\ ,,\ 2\tau + 2g'\right)$$

$$+\tfrac{33}{2}e'\left(\tfrac{1}{2}\ ,,\ 2\tau - 2g' - \tfrac{1}{2}\ ,,\ 2\tau + 2g'\right),$$

which is =

$$(-6-6=)\ -12e'\,t\cos 2\tau$$

$$+3\ ,,\ 2\tau - g'$$

$$-3\ ,,\ 2\tau + g'$$

$$\left(6+\tfrac{33}{4}=\right)\ +\tfrac{57}{4}e'\ ,,\ 2\tau - 2g'$$

$$\left(6-\tfrac{33}{4}=\right)\ -\tfrac{9}{4}e'\ ,,\ 2\tau + 2g'.$$

whence, adding the two products, and observing that $\varepsilon^2 = 1$, the required terms are,

$$=\ \tfrac{3}{2}e'\,t$$

$$+\tfrac{3}{2}\ t\cos g'$$

$$\left(\tfrac{9}{2}-12=\right)\ -\tfrac{15}{2}e'\ ,,\ 2\tau$$

$$\left(\tfrac{9}{4}+3=\right)\ +\tfrac{21}{4}\ ,,\ 2\tau - g'$$

$$\left(\tfrac{9}{4}-3=\right)\ -\tfrac{3}{4}\ ,,\ 2\tau + g'$$

$$+\tfrac{9}{2}e'\ ,,\ 2g'$$

$$\left(\tfrac{45}{4}+\tfrac{57}{4}=\right)\ +\tfrac{51}{2}e'\ ,,\ 2\tau - 2g'$$

$$\left(\tfrac{9}{4}-\tfrac{9}{4}=\right)\ 0\ ,,\ 2\tau + 2g'.$$

Annex 5.

Calculation of a part of δQ, viz.

$$\delta Q = \varrho^2 \left[\left(-\frac{3}{2} \sin 2v - 2v' \right) \delta \frac{1}{\varrho'^3} + (3 \cos 2v - 2v') \frac{\delta v'}{\varrho'^3} \right],$$

$-\frac{3}{2} \sin 2v - 2v' =$	$\delta \frac{1}{\varrho'^3} =$
$-\frac{3}{2}$ sin 2τ	$3e'$ t
$-3e'$,, $2\tau - g'$	$+3$ $t \cos g'$
$+3e'$,, $2\tau + g'$	$+9e'$,, $2g'$,

Product is

$$-\frac{9}{2} e' \quad t \sin 2\tau$$

$$-\frac{9}{2} \left(\frac{1}{2} \; t \sin 2\tau + g' \quad + \frac{1}{2} \; t \sin 2\tau - g' \right)$$

$$-9e' \left(\frac{1}{2} \quad ,, \; 2\tau \quad + \frac{1}{2} \quad ,, \; 2\tau - 2g' \right)$$

$$+9e' \left(\frac{1}{2} \quad ,, \; 2\tau + 2g' \quad + \frac{1}{2} \quad ,, \; 2\tau \right)$$

$$-\frac{27}{2} e' \left(\frac{1}{2} \quad ,, \; 2\tau + 2g' \quad + \frac{1}{2} \quad ,, \; 2\tau - 2g' \right),$$

which is =

$$\left(-\frac{9}{2} - \frac{9}{2} + \frac{9}{2} = \right) \; -\frac{9}{2} e' \; t \sin 2\tau$$

$$-\frac{9}{4} \quad ,, \; 2\tau - g'$$

$$-\frac{9}{4} \quad ,, \; 2\tau + g'$$

$$\left(-\frac{9}{2} - \frac{27}{4} = \right) \; -\frac{45}{4} e' \quad ,, \; 2\tau - 2g'$$

$$\left(+\frac{9}{2} - \frac{27}{4} = \right) \; -\frac{9}{4} e' \quad ,, \; 2\tau + 2g';$$

and

$\cos 2v - 2v' =$	$\frac{\delta v'}{\varepsilon'^3} =$
$3 \quad \cos 2\tau$	$2 \quad t \sin g'$
$+ 6e' \quad ,, \quad 2\tau - g'$	
$- 6e' \quad ,, \quad 2\tau + g'$	$+ \frac{11}{2} e' \quad ,, \quad 2g',$

Product is

$$6 \left(\tfrac{1}{2} \; t \sin 2\tau + 2g' \quad -\tfrac{1}{2} \; t \sin 2\tau - g'\right)$$
$$+ 12e' \left(\tfrac{1}{2} \quad ,, \; 2\tau \quad -\tfrac{1}{2} \quad ,, \; 2\tau - 2g'\right)$$
$$- 12e' \left(\tfrac{1}{2} \quad ,, \; 2\tau + 2g' \quad -\tfrac{1}{2} \quad ,, \; 2\tau \quad\right)$$
$$+ \tfrac{33}{2} e' \left(\tfrac{1}{2} \quad ,, \; 2\tau + 2g' \quad -\tfrac{1}{2} \quad ,, \; 2\tau - 2g'\right);$$

which is =

$$(6 + 6 =) \quad 12e' \quad t \sin 2\tau$$
$$- 3 \quad ,, \; 2\tau - g'$$
$$+ 3 \quad ,, \; 2\tau + g'$$
$$\left(-6 - \tfrac{33}{4} =\right) \quad -\tfrac{57}{4} e' \quad ,, \; 2\tau - 2g'$$
$$\left(-6 + \tfrac{33}{4} =\right) \quad +\tfrac{9}{4} e' \quad ,, \; 2\tau + 2g'.$$

Adding the two products together, and observing that $\varepsilon^2 = 1$, the required terms are

$$\left(-\tfrac{9}{2} + 12 =\right) \quad \tfrac{15}{2} e' \; t \sin 2\tau$$
$$\left(-\tfrac{9}{4} - 3 =\right) \quad -\tfrac{21}{4} \quad ,, \; 2\tau - g'$$
$$\left(-\tfrac{9}{4} + 3 =\right) \quad +\tfrac{3}{4} \quad ,, \; 2\tau + g'$$
$$\left(-\tfrac{45}{4} - \tfrac{57}{4} =\right) \quad -\tfrac{51}{2} e' \quad ,, \; 2\tau - 2g'$$
$$\left(-\tfrac{9}{4} + \tfrac{9}{4} =\right) \quad 0 \quad ,, \; 2\tau + 2g'.$$

Annex 6.

Calculation of terms in δ P, viz.

		$n^{-1}\times$
-3	$\sin 2\tau$	-3 $\cos g'$
$-\frac{21}{2}e'$	$,, \; 2\tau - g'$	$-\frac{9}{4}e'$ $,, \; 2g'$;
$+\frac{3}{2}e'$	$,, \; 2\tau + g'$	

the product of which is

$$9\left(\tfrac{1}{2}\sin 2\tau + g' + \tfrac{1}{2}\sin 2\tau - g'\right)$$
$$+\tfrac{63}{2}e'\left(\tfrac{1}{2}\,,, \; 2\tau + \tfrac{1}{2}\,,, \; 2\tau - 2g'\right)$$
$$-\tfrac{9}{2}e'\left(\tfrac{1}{2}\,,, \; 2\tau + 2g' + \tfrac{1}{2}\,,, \; 2\tau\right)$$
$$+\tfrac{27}{4}e'\left(\tfrac{1}{2}\,,, \; 2\tau + 2g' + \tfrac{1}{2}\,,, \; 2\tau - 2g'\right),$$

which is =

		$n^{-1}\times$
		$\frac{27}{2}e'$ $\sin 2\tau$
	$+\frac{9}{2}$	$,, \; 2\tau - g'$
	$+\frac{9}{2}$	$,, \; 2\tau + g'$
$\left(\frac{63}{4} - \frac{9}{4} =\right)$	$+\frac{153}{8}e'$	$,, \; 2\tau - 2g'$
$\left(-\frac{9}{4} + \frac{27}{8} =\right)$	$+\frac{9}{8}e'$	$,, \; 2\tau + 2g'$.

Annex 7.

Calculation of terms in δQ, viz.

	$n^{-1}\times$
$-3 \quad \cos 2\sigma$	$-3 \quad \cos g'$
$-\frac{21}{2}e' \quad ,, \quad 2\sigma - g'$	$-\frac{9}{4}e' \quad ,, \quad 2g';$
$+\frac{3}{2}e' \quad ,, \quad 2\sigma + g'$	

the product of which is

$$9\left(\tfrac{1}{2}\cos 2\sigma - g' + \tfrac{1}{2}\cos 2\sigma + g'\right)$$

$$+\tfrac{63}{2}e'\left(\tfrac{1}{2} \;,,\; 2\sigma - 2g' + \tfrac{1}{2} \;,,\; 2\sigma\right)$$

$$-\tfrac{9}{2}e'\left(\tfrac{1}{2} \;,,\; 2\sigma + \tfrac{1}{2} \;,,\; 2\sigma + 2g'\right)$$

$$+\tfrac{27}{4}e'\left(\tfrac{1}{2} \;,,\; 2\sigma - 2g' + \tfrac{1}{2} \;,,\; 2\sigma + 2g'\right),$$

which is $=$

	$n^{-1}\times$
$\left(\frac{63}{4} - \frac{9}{4} =\right)$	$\frac{27}{2}e' \quad \cos 2\sigma$
	$+\frac{9}{2} \quad ,, \quad 2\sigma - g'$
	$+\frac{9}{2} \quad ,, \quad 2\sigma + g'$
$\left(\frac{63}{4} + \frac{27}{8} =\right)$	$+\frac{155}{8}e' \quad ,, \quad 2\sigma - 2g'$
$\left(-\frac{9}{4} + \frac{27}{8} =\right)$	$+\frac{9}{8}e' \quad ,, \quad 2\sigma + 2g'.$

Annex 8.

Calculation of $n\int \delta Q\, dt$.

We have

$$n\int \sin n\alpha t\ dt = -\frac{1}{\alpha}\, t\cos n\alpha t + \frac{1}{n\alpha}\sin n\alpha t,$$

$$\int \cos n\alpha t\ dt = \frac{1}{n\alpha}\sin n\alpha t,$$

and in all the arguments α is taken $= 2$.

$n\int \delta Q\, dt =$	$-\frac{15}{4}\, e' t\cos 2\sigma$	$\left(+\frac{15}{8}+\frac{27}{4}=\right)$	$\overbrace{+\frac{69}{8}\, e'\sin 2\sigma}^{n^{-1}\times}$
	$+\frac{21}{8}$,, $2\sigma - g'$	$\left(-\frac{21}{16}+\frac{9}{4}=\right)$	$+\frac{15}{16}$,, $2\sigma - g'$
	$-\frac{3}{8}$,, $2\sigma + g'$	$\left(+\frac{3}{16}+\frac{9}{4}=\right)$	$+\frac{39}{16}$,, $2\sigma + g'$
	$+\frac{51}{4}\, e'$,, $2\sigma - 2g'$	$\left(-\frac{51}{8}+\frac{153}{16}=\right)$	$+\frac{51}{16}\, e'$,, $2\sigma - 2g'$
	0 ,, $2\sigma + 2g'$	$\left(0+\frac{9}{16}=\right)$	$+\frac{9}{16}$,, $2\sigma + 2g'$.

Annex 9.

Calculation of $\delta P + 2n\int \delta Q\, dt$; viz. this is

	$\frac{3}{2}\, e' t$		
	$\frac{3}{2}\ t\cos g'$		
$\left(-\frac{15}{2}-\frac{15}{2}=\right)-15\, e'$	,, 2σ	$\left(+\frac{27}{2}+\frac{69}{4}=\right)$	$\overbrace{+\frac{123}{4}\, e'\sin 2\sigma}^{n^{-1}\times}$
$\left(+\frac{21}{4}+\frac{21}{4}=\right)+\frac{21}{2}$	,, $2\sigma - g'$	$\left(+\frac{9}{2}+\frac{15}{8}=\right)+\frac{51}{8}$	,, $2\sigma - g'$
$\left(-\frac{3}{4}-\frac{3}{4}=\right)-\frac{3}{2}$	,, $2\sigma + g'$	$\left(+\frac{9}{2}+\frac{39}{8}=\right)+\frac{75}{8}$	,, $2\sigma + g'$
$\left(+\frac{9}{2}+0=\right)+\frac{9}{2}\, e'$	,, $2g'$	$\left(0+0=\right)\ 0$	,, $2g'$
$\left(+\frac{51}{2}+\frac{51}{2}=\right)+51\, e'$	,, $2\sigma - 2g'$	$\left(+\frac{153}{8}+\frac{51}{8}=\right)+\frac{51}{2}\, e'$	,, $2\sigma - 2g'$
$\left(0+0=\right)\ 0$	,, $2\sigma + 2g'$	$\left(+\frac{9}{8}+\frac{9}{8}=\right)+\frac{9}{4}\, e'$	,, $2\sigma + 2g'$.

Annex 10.

Calculation of $\delta\varrho$ from the equation

$$\frac{d^2\delta\varrho}{dt^2} + n^2 d\varrho = m^2 n^2 \left(\delta P + 2n \int \delta Q\, dt\right).$$

In $\frac{d^2\delta\varrho}{dt^2} + n^2\delta\varrho$, a term $n^2 t \cos n\alpha t$, gives in $\delta\varrho$,

$$\frac{1}{1-\alpha^2} t \cos n\alpha t + \frac{2\alpha}{(1-\alpha^2)^2}\frac{1}{n} \sin n\alpha t\,;$$

and a term $n \sin n\alpha t$, gives in $\delta\varrho$,

$$\frac{1}{1-\alpha^2}\frac{1}{n} \sin n\alpha t.$$

and $\alpha = 2$, and $\therefore \frac{1}{1-\alpha^2} = -\frac{1}{3}$, for all the args. except only $\alpha = m$ or $2m$, and $\therefore \frac{1}{1-\alpha^2} = 1$, for the args. g', $2g'$.

		$n^{-1}\times$
$\delta\varrho =$	$\frac{3}{2} m^2 e' t$	
	$+\frac{3}{2} m^2 \quad t \cos g'$	$\left(+3 \quad =\right) + 3\ m^3 \sin g'$
	$+5\ m^2 e' \quad ,, \quad 2\tau$	$\left(-\frac{20}{3}-\frac{41}{4} =\right) -\frac{203}{12} m^2 e' \quad ,, \quad 2\tau$
	$-\frac{7}{2} m^2 \quad ,, \quad 2\tau - g'$	$\left(+\frac{14}{3}-\frac{17}{8} =\right) +\frac{61}{24} m^2 \quad ,, \quad 2\tau - g'$
	$+\frac{1}{2} m^2 \quad ,, \quad 2\tau + g'$	$\left(-\frac{2}{3}-\frac{25}{8} =\right) -\frac{91}{24} m^2 \quad ,, \quad 2\tau + g'$
	$+\frac{9}{2} m^2 e' \quad ,, \quad 2g'$	$\left(+18 + 0 =\right) + 18\, m^3 e' \quad ,, \quad 2g'$
	$-17\, m^2 e' \quad ,, \quad 2\tau - 2g'$	$\left(+\frac{68}{3}+\frac{17}{2} =\right) +\frac{85}{6} m^2 e' \quad ,, \quad 2\tau - 2g'$
	$0 \quad ,, \quad 2\tau + 2g'$	$\left(\ 0 - \frac{3}{4} =\right) -\frac{3}{4} m^2 e' \quad ,, \quad 2\tau + 2g'.$

Annex 11.

Calculation of $\frac{d\delta v}{dt}$; viz. this is

$$= -2n\delta\varrho + m^2n^2\int\delta Q\,dt.$$

$\frac{d\delta v}{dt} =$ $n \times$

$\left(-3 \quad =\right) -3\ m^2e'\ t$	
$\left(-3 \quad =\right) -3\ m^2\ t\cos g'$	$\left(-6 \quad =\right) -6\ m^3\ \sin g'$
$\left(-10-\frac{15}{4} =\right) -\frac{55}{4}\ m^2e'$,, 2σ	$\left(+\frac{203}{6}+\frac{69}{8} =\right) +\frac{1019}{24}\ m^2e'$,, 2σ
$\left(+7+\frac{21}{8} =\right) +\frac{77}{8}\ m^2$,, $2\sigma-g'$	$\left(-\frac{61}{12}+\frac{15}{16} =\right) -\frac{199}{48}\ m^2$,, $2\sigma-g'$
$\left(-1-\frac{3}{8} =\right) -\frac{11}{8}\ m^2$,, $2\sigma+g'$	$\left(+\frac{91}{12}+\frac{39}{16} =\right) +\frac{481}{48}\ m^2$,, $2\sigma+g'$
$\left(-9+0 =\right) -9\ m^2e'$,, $2g'$	$\left(-36+0 =\right) -36\ m^3e'$,, $2g'$
$\left(+34+\frac{51}{4} =\right) +\frac{187}{4}m^2e'$,, $2\sigma-2g'$	$\left(-\frac{85}{3}+\frac{51}{16} =\right) -\frac{1207}{48}m^2e'$,, $2\sigma-2g'$
$\left(0+0\right) \quad 0$,, $2\sigma+2g'$	$\left(+\frac{3}{2}+\frac{9}{16} =\right) +\frac{33}{16}\ m^2e'$,, $2\sigma+2g'$.

Annex 12.

Calculation of δv from the foregoing value of $\frac{d\delta v}{dt}$.

We have

$$n\int\cos n\alpha t\ dt = \frac{1}{\alpha}\ t\ \sin n\alpha t + \frac{1}{n\alpha^2}\cos n\alpha t,$$

$$\int\sin n\alpha t\ dt = -\frac{1}{n\alpha}\cos n\alpha t;$$

$\alpha = 2$ for all the arguments, except only $\alpha = m$, $2\ m$, for the arguments g', $2\ g'$, respectively.

$\delta v =$		$-\frac{3}{2} m^2 n e' t^2 + n^{-1} \times$
$-3 m t \sin g'$	$\left(-\frac{3}{m^2} + 6 =\right)$	$-3 + 6m^2 \cos g'$
$-\frac{55}{8} m^2 e'$,, 2τ	$\left(-\frac{55}{16} - \frac{1019}{48} =\right)$	$-\frac{74}{3} m^2 e'$,, 2τ
$+\frac{77}{16} m^2$,, $2\tau - g'$	$\left(+\frac{77}{32} + \frac{199}{96} =\right)$	$+\frac{215}{48} m^2$,, $2\tau - g'$
$-\frac{11}{16} m^2$,, $2\tau + g'$	$\left(-\frac{11}{32} - \frac{481}{96} =\right)$	$-\frac{257}{48} m^2$,, $2\tau + g'$
$-\frac{9}{2} m e'$,, $2g'$	$\left(-\frac{9}{4m^2} + 18 =\right)$	$\left(-\frac{9}{4} + 18 m^2\right) e'$,, $2g'$
$+\frac{187}{8} m^2 e'$,, $2\tau - 2g'$	$\left(+\frac{187}{16} + \frac{1207}{96} =\right)$	$+\frac{2329}{96} m^2 e'$,, $2\tau - 2g'$
0 ,, $2\tau + 2g'$	$\left(0 - \frac{33}{32} =\right)$	$-\frac{33}{32} m^2 e'$,, $2\tau + 2g'$.

The remaining Annexes relate to the determination of the non-periodic or secular terms of the order m^4, in $\delta \varrho$ and δv respectively.

Annex 13.

Calculation of term of $\left(n^2 + \frac{2n^2}{\varrho^3} - 3\left(\frac{dv}{dt}\right)^2\right) \delta \varrho$.

We have

$n^2 + \frac{2n^2}{\varrho^3} - 3\left(\frac{dv}{dt}\right)^2 =$		$n^2 \times$	$\delta \varrho =$
$\left(1 + 2 + m^2 - 3 =\right)$	m^2		$\frac{3}{2} m^2 e' t$
$\left(-9 + 18 =\right)$	$+9 m^2 e' \cos g'$		$+\frac{3}{2} m^2 t \cos g'$
$\left(+6 - \frac{33}{2} =\right)$	$-\frac{21}{2} m^2$,, 2τ		$+5 m^2 e'$,, 2τ
$\left(+21 - \frac{231}{4} =\right)$	$-\frac{147}{4} m^2 e'$,, $2\tau - g'$		$-\frac{7}{2} m^2$,, $2\tau - g'$
$\left(-3 + \frac{33}{4} =\right)$	$+\frac{21}{4} m^2 e'$,, $2\tau + g'$		$+\frac{1}{2} m^2$,, $2\tau + g'$
			+ &c.

where, in the second factor, the arguments not occurring in the first factor are omitted, as not giving rise to any non-periodic

term; and so in other similar cases. Hence term of product is

$$m^4 n^2 e' t \quad 1 \cdot \frac{3}{2}$$

$$+\frac{1}{2} \cdot \quad 9 \cdot \frac{3}{2}$$

$$+\frac{1}{2} \cdot -\frac{21}{2} \cdot \quad 5$$

$$+\frac{1}{2} \cdot -\frac{147}{4} \cdot -\frac{7}{2}$$

$$+\frac{1}{2} \cdot \frac{21}{4} \cdot \frac{1}{2}$$

$$= \left(\frac{3}{2} + \frac{27}{4} - \frac{105}{4} + \frac{1029}{16} + \frac{21}{16} = \right) + \frac{381}{8} m^4 n^2 e' t.$$

Annex 14.

Calculation of term of $m^2 n^2 \delta \mathrm{P}$.

$m^2 n^2 \delta \mathrm{P}$, the part involving δv is

$\underbrace{m^2 n^2 \times}$	$\delta v =$
	$-3 \; m \quad t \sin g'$
$-3 \quad \sin 2\tau$	$-\frac{55}{8} m^2 e' \;\; ,, \;\; 2\tau$
$-\frac{21}{2} e' \;\; ,, \;\; 2\tau - g'$	$+\frac{77}{16} m^2 \quad ,, \;\; 2\tau - g'$
$+\frac{3}{2} e' \;\; ,, \;\; 2\tau + g'$	$-\frac{11}{16} m^2 \quad ,, \;\; 2\tau + g'$
	+ &c.

and term of the product is

$$m^4 n^2 e' t \quad \frac{1}{2} \cdot -3 \cdot -\frac{55}{8}$$

$$+\frac{1}{2} \quad -\frac{21}{2} \cdot \quad \frac{77}{16}$$

$$+\frac{1}{2} \cdot \quad \frac{3}{2} \cdot -\frac{11}{16}$$

$$= \left(\frac{165}{16} - \frac{1617}{64} - \frac{33}{64} = \right) - \frac{495}{32} m^4 n^2 e' t.$$

Annex 15.

Calculation of term of $m^2 n^2 \delta P$.

$m^2 n^2 \delta P$, the term involving $\delta \varrho$ is

$m^2 n^2 \times$	$\delta \varrho =$
$\frac{1}{2}$	$\frac{3}{2} m^2 e' \ t$
$+ \frac{3}{2} e' \cos g'$	$+ \frac{3}{2} m^2 \quad t \cos g'$
$+ \frac{3}{2} \quad ,, \ 2\tau$	$+ 5 \, m^2 e' \quad ,, \ 2\tau$
$+ \frac{21}{4} e' \quad ,, \ 2\tau - g'$	$- \frac{7}{2} m^2 \quad ,, \ 2\tau - g'$
$- \frac{3}{4} e' \quad ,, \ 2\tau + g'$	$+ \frac{1}{2} m^2 \quad ,, \ 2\tau + g'$
	+ &c.

and term of the product is

$$m^4 n^2 e' t \quad \frac{1}{2} \cdot \frac{3}{2}$$

$$+ \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{2}$$

$$+ \frac{1}{2} \cdot \frac{3}{2} \cdot 5$$

$$+ \frac{1}{2} \cdot \frac{21}{4} \cdot -\frac{7}{2}$$

$$+ \frac{1}{2} \cdot -\frac{3}{4} \cdot \frac{1}{2}$$

$$= \left(\frac{3}{4} + \frac{9}{8} + \frac{15}{4} - \frac{147}{16} - \frac{3}{16} = \right) - \frac{15}{4} m^4 n^2 e' t .$$

Annex 16.

Calculation of $\overset{\cos}{\sin}\ 2v - 2v'$, as far as m^2.

$$\cos 2v - 2v' = \cos 2\tau + X = \cos 2\tau$$
$$- X \sin 2\tau$$
$$- \tfrac{1}{2} X^2 \cos 2\tau,$$

$$\sin 2v - 2v' = \sin 2\tau + X = \sin 2\tau$$
$$+ X \cos 2\tau$$
$$- \tfrac{1}{2} X^2 \sin 2\tau,$$

where

$$X = -\left(4 + 6m\right)e' \quad \sin g'$$
$$+ \tfrac{11}{4} m^2 \quad ,, \ 2\tau$$
$$+ \tfrac{77}{8} m^2 e' \quad ,, \ 2\tau - g'$$
$$- \tfrac{11}{8} m^2 e' \quad ,, \ 2\tau + g',$$

and thence

$$X \sin 2\tau = -\left(4 + 6m\right)e' \left(\tfrac{1}{2}\cos 2\tau - g' \ -\tfrac{1}{2}\cos 2\tau + g'\right)$$
$$+ \tfrac{11}{4} m^2 \left(\tfrac{1}{2} \quad -\tfrac{1}{2} \ ,, \ 4\tau \quad\right)$$
$$+ \tfrac{77}{8} m^2 e' \left(\tfrac{1}{2}\cos g' \quad -\tfrac{1}{2} \ ,, \ 4\tau - g'\right)$$
$$- \tfrac{11}{8} m^2 e' \left(\tfrac{1}{2} \ ,, \ g' \quad -\tfrac{1}{2} \ ,, \ 4\tau + g'\right),$$

which is =

$$\frac{11}{8}m^2$$

$$\left(\frac{77}{16} - \frac{11}{16} =\right) + \frac{33}{8}m^2 e' \quad \cos g'$$

$$-\left(2+3m\right)e' \quad ,, \quad 2\tau - g'$$

$$+\left(2+3m\right)e' \quad ,, \quad 2\tau + g'$$

$$-\frac{11}{8}m^2 \quad ,, \quad 4\tau$$

$$-\frac{77}{16}m^2 e' \quad ,, \quad 4\tau - g'$$

$$+\frac{11}{16}m^2 e' \quad ,, \quad 4\tau + g';$$

$$\text{X}\cos 2\tau = -\left(4+6m\right)e' \quad \left(\frac{1}{2}\sin 2\tau + g' \quad -\frac{1}{2}\sin 2\tau - g'\right)$$

$$+\frac{11}{4}m^2 \quad \left(\frac{1}{2} \quad ,, \quad 4\tau \qquad\qquad\right)$$

$$+\frac{77}{8}m^2 e' \quad \left(\frac{1}{2} \quad ,, \quad 4\tau - g' \quad -\frac{1}{2} \quad ,, \quad g' \quad\right)$$

$$-\frac{11}{8}m^2 e' \quad \left(\frac{1}{2} \quad ,, \quad 4\tau + g' \quad -\frac{1}{2} \quad ,, \quad g' \quad\right),$$

which is =

$$\left(-\frac{77}{16} - \frac{11}{16} =\right) -\frac{11}{2}m^2 e' \quad \sin g'$$

$$+\left(2+3m\right)e' \quad ,, \quad 2\tau - g'$$

$$-\left(2+3m\right)e' \quad ,, \quad 2\tau + g'$$

$$+\frac{11}{8}m^2 \quad ,, \quad 4\tau$$

$$+\frac{77}{16}m^2 e' \quad ,, \quad 4\tau - g'$$

$$-\frac{11}{16}m^2 e' \quad ,, \quad 4\tau + g';$$

we have, moreover,

$$X^2 = -2\left(4+6m\right)e'\sin g' \,.\, \frac{11}{4}m^2\sin 2\tau$$

$$= -22\,m^2e'\left(\frac{1}{2}\cos 2\tau - g' \quad -\frac{1}{2}\cos 2\tau + g'\right)$$

$$= -11\,m^2e' \quad \cos 2\tau - g'$$

$$+11\,m^2e' \quad ,, \quad 2\tau + g',$$

and thence

$$X^2\cos 2\tau = -11\,m^2e'\left(\frac{1}{2}\cos 4\tau - g' + \frac{1}{2}\cos g'\right)$$

$$+11\,m^2e'\left(\frac{1}{2} \;,, \; 4\tau + g' + \frac{1}{2} \;,, \; g'\right)$$

$$= \left(-\frac{11}{2}+\frac{11}{2}=\right) \quad 0 \qquad \cos g'$$

$$-\frac{11}{2}m^2e' \quad ,, \quad 4\tau - g'$$

$$+\frac{11}{2}m^2e' \quad ,, \quad 4\tau + g';$$

and

$$X^2\sin 2\tau = -11\,m^2e'\left(\frac{1}{2}\sin 4\tau - g' + \frac{1}{2}\sin g'\right)$$

$$-11\,m^2e'\left(\frac{1}{2} \;,, \; 4\tau + g' - \frac{1}{2} \;,, \; g'\right)$$

$$= \left(-\frac{11}{2}-\frac{11}{2}=\right) -11\,m^2e' \quad \sin g'$$

$$-\frac{11}{2}m^2e' \quad ,, \quad 4\tau - g'$$

$$+\frac{11}{2}m^2e' \quad ,, \quad 4\tau + g';$$

and thence

$$\begin{aligned}
\cos 2v - 2v' = & \quad - \tfrac{11}{8} m^2 \\
& + 1 \qquad \cos 2\tau \\
& - \tfrac{33}{8} m^2 e' \quad ,, \; g' \\
& + \left(2 + 3m\right) e' \quad ,, \; 2\tau - g' \\
& - \left(2 + 3m\right) e' \quad ,, \; 2\tau + g' \\
& + \tfrac{11}{8} m^2 \quad ,, \; 4\tau \\
\left(+ \tfrac{77}{16} + \tfrac{11}{4} = \right) & + \tfrac{121}{16} m^2 e' \quad ,, \; 4\tau - g' \\
\left(- \tfrac{11}{16} - \tfrac{11}{4} = \right) & - \tfrac{55}{16} m^2 e' \quad ,, \; 4\tau + g';
\end{aligned}$$

and

$$\begin{aligned}
\sin 2v - 2v' = & \qquad 1 \qquad \sin 2\tau \\
\left(- \tfrac{11}{2} + \tfrac{11}{2} = \right) & \qquad 0 \quad m^2 e' \quad ,, \; g' \\
& + \left(2 + 3m\right) e' \quad ,, \; 2\tau - g' \\
& - \left(2 + 3m\right) e' \quad ,, \; 2\tau + g' \\
& + \tfrac{11}{8} m^2 \quad ,, \; 4\tau \\
\left(\tfrac{77}{16} + \tfrac{11}{4} = \right) & + \tfrac{121}{16} m^2 e' \quad ,, \; 4\tau - g' \\
\left(- \tfrac{11}{16} - \tfrac{11}{4} = \right) & - \tfrac{55}{16} m^2 e' \quad ,, \; 4\tau + g'.
\end{aligned}$$

Annex 17.

Calculation of term in δQ.

The part of δQ containing δv is $\frac{\varrho^2}{\varrho'^3}\left(-3\cos 2v - 2v'\right)\delta v$; and it is necessary to find in $\frac{\varrho^2}{\varrho'^3}\left(-3\cos 2v - 2v'\right)$ the coefficient of $\cos g'$ as far as m^2; this is, in fact, required *post* Annex 21, for the calculation of $m^2 n^2 (C + \int \delta Q\, dt)$.

$\varrho^2 =$	$\frac{1}{\varrho'^3} =$
$1 - \frac{1}{3}m^2$	1
$+ 3m^2 e' \cos g'$	$+ 3e' \cos g'$
$- 2m^2 \quad ,, \; 2\tau$	
$- 7m^2 e' \quad ,, \; 2\tau - g'$	
$+ 1m^2 e' \quad ,, \; 2\tau + g'$	

and thence

$\frac{\varrho^2}{\varrho'^3} =$	$-3\cos 2v - 2v' =$
$1 - \frac{1}{3}m^2$	$\frac{33}{8}m^2$
$+ \left(3 + 2m^2\right)e' \cos g'$	$+ \frac{99}{8}m^2 e' \cos g'$
$- 2m^2 \quad ,, \; 2\tau$	$- 3 \quad ,, \; 2\tau$
$\left(-7 - \frac{1}{2}.2.3 =\right) - 10m^2 e' \quad ,, \; 2\tau - g'$	$-\left(6 + 9m\right)e' \quad ,, \; 2\tau - g'$
$\left(+1 - \frac{1}{2}.2.3 =\right) - 2m^2 e' \quad ,, \; 2\tau + g'$	$+\left(6 + 9m\right)e' \quad ,, \; 2\tau + g'$
	&c.

where, in the second column, the omitted terms have arguments containing 4τ, and, consequently, do not, by combination with the first column, give rise to any term with the argument g'. The term arg. g' arises from the combinations

$$\frac{33}{8} m^2 \quad .3\, e' \quad \cos g'$$

$$+ \frac{99}{8} m^2 e' .1 \qquad ,, \quad g'$$

$$- 3 \quad . - 10\, m^2 e' \left(\cos 2\tau \cos 2\tau - g' = \frac{1}{2} \cos 4\tau - g' + \frac{1}{2} \cos g'\right)$$

$$- 3 \quad . - 2\, m^2 e' \left(\,,, \; 2\tau \;\, ,, \;\, 2\tau + g' = \frac{1}{2} \;\, ,, \;\, 4\tau + g' + \frac{1}{2} \;\, ,, \;\, g'\right)$$

$$- 6 e' . - 2\, m^2 \quad \left(\,,, \; 2\tau \;\, ,, \;\, 2\tau - g' = \frac{1}{2} \;\, ,, \;\, 4\tau - g' + \frac{1}{2} \;\, ,, \;\, g'\right)$$

$$+ 6 e' . - 2\, m^2 \quad \left(\,,, \; 2\tau \;\, ,, \;\, 2\tau + g' = \frac{1}{2} \;\, ,, \;\, 4\tau + g' + \frac{1}{2} \;\, ,, \;\, g'\right)$$

so that the required term is

$$\left(\frac{99}{8} + \frac{99}{8} + 15 + 3 + 6 - 6 = \right) + \frac{171}{4} m^2 e' \cos g'.$$

and annexing this to the terms found Annex 2, the part of δQ which contains δv is

$$\left.\begin{array}{lll} \frac{171}{4} m^2 e' & \cos g' & \\ - \;\; 3 & ,, \;\, 2\tau & \\ - \;\; \frac{21}{2} \; e' & ,, \;\, 2\tau - g' & \delta v. \\ + \;\; \frac{3}{2} \; e' & ,, \;\, 2\tau + g' & \end{array}\right|$$

Annex 18.

Calculation of term of $m^2 n^2 \delta P$.

The part of δP involving $\delta v'$ and $\delta \varrho$ is

$$\delta P = \varrho \left[\left(\frac{1}{2} + \frac{3}{2} \cos 2v - 2v'\right) \delta \frac{1}{\varrho'^3} + \left(3 \sin 2v - 2v'\right) \frac{\delta v'}{\varrho'^3}\right];$$

$\varrho =$	$\frac{1}{2} + \frac{3}{2}\cos 2v - 2v' =$	$\delta \frac{1}{\varrho^3} =$
$1 - \frac{1}{6}m^2$	$\frac{1}{2} - \frac{33}{16}m^2$	$3e'\ t$
$+ \frac{3}{2}m^2 e' \cos g'$	$- \frac{99}{16}m^2 e' \quad \cos g'$	$+ 3 \quad t \cos 2g'$
$- \quad m^2 \quad ,, \ 2\tau$	$+ \frac{3}{2} \quad ,, \ 2\tau$	$+ 9e' \quad ,, \ 2g'$
$- \frac{7}{2}m^2 e' \ ,, \ 2\tau - g'$	$+ \left(3 + \frac{9}{2}m\right)e' \ ,, \ 2\tau - g'$	
$+ \frac{1}{2}m^2 e' \ ,, \ 2\tau + g'$	$- \left(3 + \frac{9}{2}m\right)e' \ ,, \ 2\tau + g'$	
	$+$ &c.	

where, in the second factor, the terms belonging to the arguments which contain 4τ (*i. e.* the arguments 4τ, $4\tau - g'$, $4\tau + g'$) are omitted. In fact, the terms in question would, in the product of the second and third factors, give rise to terms with arguments containing 4τ, and as there are no such terms in the first factor, there is no resulting secular term.

The product of the second and third factors is

$$\left(\frac{3}{2} - \frac{99}{16}m^2\right)e' \quad t$$

$$+ \left(\frac{3}{2} - \frac{99}{16}m^2\right) \quad t \cos g'$$

$$+ \frac{9}{2}e' \quad ,, \ 2\tau$$

$$-\frac{297}{16}m^2 e' \left(\frac{1}{2}t \quad + \frac{1}{2}t \cos 2g' \right)$$

$$+ \frac{9}{2} \left(\frac{1}{2}t \cos 2\tau - g' \quad + \frac{1}{2} \ ,, \ 2\tau + g'\right)$$

$$+ \left(9 + \frac{27}{2}m\right)e' \left(\frac{1}{2}t \ ,, \ 2\tau - 2g' \quad + \frac{1}{2} \ ,, \ 2\tau \right)$$

$$- \left(9 + \frac{27}{2}m\right)e' \left(\frac{1}{2} \ ,, \ 2\tau + 2g' \quad + \frac{1}{2} \ ,, \ 2\tau \right)$$

$$+ \left(\frac{9}{2} - \frac{297}{16}m^2\right)e' \quad t \cos 2g'$$

$$+ \frac{27}{2}e' \left(\frac{1}{2}t \cos 2\tau - 2g' \quad + \frac{1}{2}t \cos 2\tau + 2g'\right),$$

and we may in this product omit the terms with arguments containing $2g'$, since the first factor does not contain any such term. The product then is

$$\left(-\frac{99}{16}-\frac{297}{32}=-\frac{495}{32}\right) \qquad \left(\frac{3}{2}-\frac{495}{32}m^2\right) e' \quad t$$

$$+\left(\frac{3}{2}-\frac{99}{16}m^2\right) \quad t\cos g'$$

$$\left(\left(\frac{9}{2}+\frac{9}{2}-\frac{9}{2}\right)e'+\left(\frac{27}{4}-\frac{27}{4}\right)me'\right) \qquad +\frac{9}{2}e' \quad ,, \quad 2\sigma$$

$$+\frac{9}{4} \quad ,, \quad 2\sigma-g'$$

$$+\frac{9}{4} \quad ,, \quad 2\sigma+g',$$

which is to be multiplied by the first factor, ϱ, and the whole by the factor m^2n^2.

The term in the product is

$$m^2n^2e't \qquad \left(\frac{3}{2}-\frac{495}{32}m^2\right)\left(1-\frac{1}{6}m^2\right)$$

$$+\frac{1}{2}\cdot\frac{3}{2} \quad . \quad \frac{3}{2}m^2$$

$$+\frac{1}{2}\cdot\frac{9}{2} \quad . \quad -m^2$$

$$+\frac{1}{2}\cdot\frac{9}{4} \quad . \quad -\frac{7}{2}m^2$$

$$+\frac{1}{2}\cdot\frac{9}{4} \quad . \quad \frac{1}{2}m^2,$$

giving the term $\frac{3}{2}m^2n^2e't$, which was found above, Annex 9, and the new terms

$$\left(-\frac{495}{32}-\frac{1}{4}+\frac{9}{8}-\frac{9}{4}-\frac{63}{16}+\frac{9}{16}=\right)-\frac{647}{32}m^4n^2e't.$$

The term of the part containing $\delta v'$ is found to be $=0$; In fact, we have

$\varrho =$	$3 \sin 2v - 2v' =$	$\frac{\delta v'}{\varrho'^3} =$
$1 - \frac{1}{6} m^2$	$3 \quad \sin 2\tau$	$2 \quad t \sin g'$
$+ \frac{3}{2} m^2 e' \cos g'$	$+ \left(6 + 9m\right) e' \;,, \; 2\tau - g'$	$+ \frac{11}{2} e' \quad ,, \; 2g'$
$- \quad m^2 \quad ,, \; 2\tau$	$- \left(6 + 9m\right) e' \;,, \; 2\tau + g'$	
$- \frac{7}{2} m^2 e' \;,, \; 2\tau - g'$	&c.	
$+ \frac{1}{2} m^2 e' \;,, \; 2\tau + g'$		

where in the second factor the terms with arguments containing 4τ are for the before-mentioned reason omitted.

The product of the second and third factors is

$$6 \left(\frac{1}{2} t \cos 2\tau - g' \quad - \frac{1}{2} t \cos 2\tau + g' \right)$$

$$+ \left(12 + 18m\right) e' \left(\frac{1}{2} \;,, \; 2\tau - 2g' - \frac{1}{2} \;,, \; 2\tau \right)$$

$$- \left(12 + 18m\right) e' \left(\frac{1}{2} \;,, \; 2\tau \quad - \frac{1}{2} \;,, \; 2\tau + 2g' \right)$$

$$+ \frac{33}{2} e' \quad \left(\frac{1}{2} \;,, \; 2\tau - 2g' - \frac{1}{2} \;,, \; 2\tau + 2g' \right)$$

which, omitting the terms with arguments containing $2g'$, is

$$- (6 + 9m + 6 + 9m) = - (12 + 18m) e' \; t \cos 2\tau$$

$$+ 3 \quad ,, \; 2\tau - g'$$

$$- 3 \quad ,, \; 2\tau + g'$$

which is to be multiplied by the first factor, ϱ, and the whole by $m^2 n^2$. The term is

$$m^4 n^2 e' t \quad \frac{1}{2} \cdot - 12 \cdot - 1$$

$$+ \frac{1}{2} \cdot \quad 3 \cdot - \frac{7}{2}$$

$$+ \frac{1}{2} \cdot - 3 \cdot \quad \frac{1}{2}$$

which is

$$\left(6 - \frac{21}{4} - \frac{3}{4} =\right) \circ \quad m^4 n^2 e' t.$$

Hence the entire term in question is the before-mentioned value

$$-\frac{647}{32} m^4 n^2 e' t.$$

Annex 19.

Calculation of term in $m^2 n^2 \frac{2}{\varrho} \frac{dv}{dt} \left(C + \int \delta Q \, dt\right)$.

We have

$\frac{2}{\varrho} \frac{dv}{dt} =$

$n \times$

$2 + \frac{1}{3} m^2$

$- 9 \, m^2 e' \cos g'$

$+ \frac{15}{2} m^2 \quad ,, \; 2\tau$

$+ \frac{105}{4} m^2 e' \quad ,, \; 2\tau - g'$

$- \frac{15}{4} m^2 e' \quad ,, \; 2\tau + g'$

See *post* Annex 20.

$m^2 n^2 \left(C + \int \delta Q \, dt\right) =$

$n \times$

$-\frac{1455}{32} m^4 e' t$

$\circ \qquad t \cos g'$

$-\frac{15}{4} m^2 \quad ,, \; 2\tau$

$+\frac{21}{8} m^2 \quad ,, \; 2\tau - g'$

$-\frac{3}{8} m^2 \quad ,, \; 2\tau + g'$

For the term $-\frac{1455}{32} m^4 e' t$ see *post* Annex 21.

The term in the product therefore is

$$m^4 n^2 e' t \qquad 2 \; . \; -\frac{1455}{32}$$

$$+\frac{1}{2} \cdot \frac{15}{2} \cdot -\frac{15}{4}$$

$$+\frac{1}{2} \cdot \frac{105}{4} \cdot -\frac{21}{8}$$

$$+\frac{1}{2} \cdot -\frac{15}{4} \cdot -\frac{3}{8}$$

which is

$$= \left(2 \cdot - \frac{1455}{32} - \frac{225}{16} + \frac{2205}{64} + \frac{45}{64}\right) m^4 n^2 e' t$$

$$= \left(2 \cdot - \frac{1455}{32} + \frac{675}{32} =\right) \quad - \frac{2235}{32} m^4 n^2 e' t.$$

Annex 20.

Calculation of $\frac{2}{\varrho} \frac{dv}{dt}$.

We have

$\frac{dv}{dt} =$	$\frac{2}{\varrho} =$
$n \times$	
1	$2 + \frac{1}{3} m^2$
$- 3\ m^2 e' \cos g'$	$- 3\ m^2 e' \cos g'$
$+ \frac{11}{4} m^2 e'$,, 2τ	$+ 2\ m^2$,, 2τ
$+ \frac{77}{8} m^2 e'$,, $2\tau - g'$	$+ 7\ m^2 e'$,, $2\tau - g'$
$- \frac{11}{8} m^2 e'$,, $2\tau + g'$	$- 1\ m^2 e'$,, $2\tau + g'$

so that the product is

$n \times$

$2 + \frac{1}{3} m^2$

$$\left(- 6 - 3 =\right) \quad - 9\ m^2 e' \cos g'$$

$$\left(+ \frac{11}{2} + 1 =\right) \quad + \frac{15}{2} m^2 \quad ,, 2\tau$$

$$\left(+ \frac{77}{4} + 7 =\right) \quad + \frac{105}{4} m^2 e' \quad ,, 2\tau - g'$$

$$\left(- \frac{11}{4} - 1 =\right) \quad - \frac{15}{8} m^2 e' \quad ,, 2\tau + g'.$$

Annex 21.

Calculation of a term in $m^2 n^2 (C + \int \delta Q \, dt)$.

The part $\frac{\varrho}{\varrho'^3}\left(-3 \sin 2v - 2v'\right)$ of δQ gives

$\frac{\varrho}{\varrho'^3}\left(-3 \sin 2v - 2v'\right) =$	$\delta\varrho =$ $\overbrace{3\ m^3\ \sin g'}^{n^{-1}\times}$
$-3 \quad \sin 2\tau$	$-\frac{203}{12} m^2 e' \quad ,, \ 2\tau$
$-\frac{21}{2} e' \quad ,, \ 2\tau - g$	$+\frac{61}{24} m^2 \quad ,, \ 2\tau - g'$
$+\frac{3}{2} e' \quad ,, \ 2\tau + g'$	$-\frac{91}{24} m^2 \quad ,, \ 2\tau + g'$

and we have thence in $m^2 n^2 \delta Q$ the term

$$m^4 n e' \quad \frac{1}{2} \cdot -3 \cdot -\frac{203}{12}$$
$$+\frac{1}{2} \cdot -\frac{21}{2} \cdot \frac{61}{24}$$
$$+\frac{1}{2} \cdot \frac{3}{2} \cdot -\frac{91}{24};$$

that is

$$\left(\frac{203}{8} - \frac{427}{32} - \frac{91}{32} =\right) + \frac{147}{16} m^4 n e'.$$

The part $\frac{\varrho^2}{\varrho'^3}\left(3 \cos 2v - 2v'\right)$ of δQ gives

$\frac{\varrho^2}{\varrho'^3}(-3 \cos 2v - 2v') =$	$\delta v =$ $\overbrace{-3 \qquad \cos g'}^{n^{-1}}$
$\frac{171}{4} m^2 e' \cos g'$	
$-3 \quad ,, \ 2\tau$	$-\frac{74}{3} m^2 e' \quad ,, \ 2\tau$
$-\frac{21}{2} e' \quad ,, \ 2\tau - g'$	$+\frac{215}{48} m^2 \quad ,, \ 2\tau - g'$
$+\frac{3}{2} e' \quad ,, \ 2\tau + g'$	$-\frac{257}{48} m^2 \quad ,, \ 2\tau + g'$

(*Ante*, Annex 17).

and we have thence in $m^2 n^2 \delta Q$ the term

$$m^4 n e' \quad \frac{1}{2} \cdot \frac{171}{4} \cdot -3$$
$$+\frac{1}{2} \cdot -3 \cdot -\frac{74}{3}$$
$$+\frac{1}{2} \cdot -\frac{21}{2} \cdot \frac{215}{48}$$
$$+\frac{1}{2} \cdot \frac{3}{2} \cdot -\frac{257}{48}$$

that is

$$\left(-\frac{513}{8} + 37 - \frac{1505}{64} - \frac{257}{64} =\right) - \frac{1749}{32} m^4 n e';$$

and, combining this with the other term just obtained, the two together are

$$\left(\frac{147}{16} - \frac{1749}{32} =\right) - \frac{1455}{32} m^4 n e';$$

and this term in $m^2 n^2 \delta Q$ gives in $m^2 n^2 (C + \int \delta Q \, dt)$ the term

$$-\frac{1455}{32} m^4 n e' t.$$

Annex 22.

Calculation of term in $-\frac{2}{\varrho}\frac{dv}{dt}\delta\varrho$.

We have

$-\frac{2}{\varrho}\frac{dv}{dt}$ (see Annex 19) =	$\varrho =$
$n \times$	
$-2 - \frac{1}{3} m^2$	$\left(\frac{3}{2} m^2 - \frac{1973}{32} m^4\right) e' t$
$+ 9 m^2 e' \cos g'$	$+ \frac{3}{2} m^2 \quad t \cos g'$
$-\frac{15}{2} m^2 \quad ,, 2\tau$	$+ 5 m^2 e' \quad ,, 2\tau$
$-\frac{105}{4} m^2 e' \quad ,, 2\tau - g'$	$-\frac{7}{2} m^2 \quad ,, 2\tau - g'$
$+\frac{15}{4} m^2 e' \quad ,, 2\tau + g'$	$+\frac{1}{2} m^2 \quad ,, 2\tau + g'$

giving, besides the term $\frac{3}{2} m^2 n e' t$ already taken account of, the term

$$
\begin{aligned}
m^4 n e' t \qquad & + \frac{1973}{16} \cdot -\frac{1}{3} \cdot \frac{3}{2} \\
& + \frac{1}{2} \cdot \quad 9 \cdot \frac{3}{2} \\
& + \frac{1}{2} \cdot -\frac{15}{2} \cdot \quad 5 \\
& + \frac{1}{2} \cdot -\frac{105}{4} \cdot -\frac{7}{2} \\
& + \frac{1}{2} \cdot \frac{15}{4} \cdot \frac{1}{2}
\end{aligned}
$$

which is

$$
\begin{aligned}
&= \left(\frac{1973}{16} - \frac{1}{2} + \frac{27}{4} - \frac{75}{4} + \frac{735}{16} - \frac{15}{16}\right) m^4 n^2 e' t \\
&= \left(\frac{1973}{16} + \frac{275}{8} =\right) \frac{2523}{16} m^4 n^2 e' t.
\end{aligned}
$$

Annex 23.

Calculation of term in $\frac{m^2 n^2}{\varrho^2} (C + \int \delta Q \, dt)$.

We have

$\frac{1}{\varrho^2} =$	$m^2 n^2 (C + \int \delta Q \, dt) =$
	$n \times$
$1 + \frac{1}{3} m^2$	$-\frac{1455}{32} m^2 e' \; t$
$- 3 \, m^2 e' \cos g'$	
$+ 2 \, m^2 \quad ,, \; 2\tau$	$-\frac{15}{4} m^2 e' \; t \cos 2\tau$
$+ 7 \, m^2 e' \quad ,, \; 2\tau - g'$	$+\frac{21}{8} m^2 e' \quad ,, \; 2\tau - g'$
$- 1 \, m^2 e' \quad ,, \; 2\tau + g'$	$-\frac{3}{8} m^2 e' \quad ,, \; 2\tau + g'$

giving the term

$$
\begin{aligned}
m^4 n e' t \qquad & - \frac{1455}{32} \\
& + \frac{1}{2} \cdot \quad 2 \cdot -\frac{15}{4} \\
& + \frac{1}{2} \cdot \quad 7 \cdot \frac{21}{8} \\
& + \frac{1}{2} \cdot -1 \cdot -\frac{3}{8}
\end{aligned}
$$

which is

$$
= \left(-\frac{1455}{32} - \frac{15}{4} + \frac{147}{16} + \frac{3}{16}\right) = \left(-\frac{1455}{32} + \frac{45}{8} =\right) -\frac{1275}{32} m^4 n e' t,
$$

and this completes the series of calculations.

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